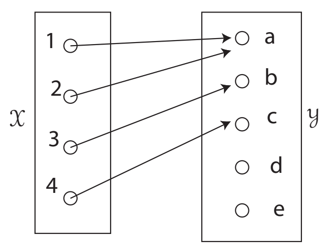
Statistics 330b/600b, Math 330b spring 2017
Homework # 1
Due: Thursday 26 January

Please attempt at least the starred problems. Please explain your reasoning. Please look at the handout `latex.pdf` *for an explanation of why solutions consisting of a long string of sequence of* $\therefore$*'s and* $\because$*'s can be hard to follow.*

*[1] Suppose T maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For each $B \subseteq \mathcal{Y}$ and $A \subseteq \mathcal{X}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$ and $T(A) := \{T(x) : x \in A\}$. In class I asserted that

$$(i) \quad T^{-1}\big(\cup_i B_i\big) = \cup_i T^{-1}(B_i)$$
$$(ii) \quad T^{-1}\big(\cap_i B_i\big) = \cap_i T^{-1}(B_i)$$
$$(iii) \quad T^{-1}\big(B^c\big) = \big(T^{-1}(B)\big)^c$$



In my experience, many students also believe that

$$(i) \quad T\big(\cup_i A_i\big) = \cup_i T(A_i)$$
$$(ii) \quad T\big(\cap_i A_i\big) = \cap_i T(A_i)$$
$$(iii) \quad T\big(A^c\big) = \big(T(A)\big)^c$$
$$(iv) \quad T^{-1}\big(T(A)\big) = A$$
$$(v) \quad T\big(T^{-1}(B)\big) = B.$$

In general, some of these assertions are false. Provide counterexamples for each of the false assertions. Maybe you could also give extra conditions under which the assertions are true. (Hint: All the counterexamples can be constructed using the special case shown in the picture.)

*[2] Let $\mathcal{G}$ denote the set of all open subsets of $\mathbb{R}^2$ and $\mathcal{H}$ denote the set of all closed half-spaces in $\mathbb{R}^2$, that is, sets of the form $\{(x,y) \in \mathbb{R}^2 : ax+by \le c\}$ for constants a, b, c. Show that $\sigma(\mathcal{G}) = \sigma(\mathcal{H})$.

[3] Suppose $\mathcal{X} = \{1,2,3,4,5\}$ and $\mathcal{E} = \{\{1,2\},\{2,3\}\}$. Define

$$\mathcal{A} = \{A \in \sigma(\mathcal{E}) : \text{ either } \{4,5\} \subseteq A \text{ or } \{4,5\} \subseteq A^c\}.$$

(i) Show that $\mathcal{A}$ is a sigma-field with $\mathcal{E} \subseteq \mathcal{A}$.

(ii) Explain why $\sigma(\mathcal{E})$ consists of exactly 16 subsets of $\mathcal{X}$.

(iii) [alternative to (i) and (ii)] Suppose $\mathcal{E}$ is a collection of subsets of some $\mathcal{X}$ and a and b are two distinct points of $\mathcal{X}$. Suppose $\mathbb{1}_E(a) = \mathbb{1}_E(b)$ for all $E \in \mathcal{E}$. Show that $\mathbb{1}_A(a) = \mathbb{1}_A(b)$ for all $A \in \sigma(\mathcal{E})$.