Statistics 330b/600b, Math 330b spring 2017
Homework # 10
Due: Thursday 13 April

*[1] Suppose $\{\mathcal{F}_n : n \in \mathbb{N}_0\}$ is a filtration defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{X_n : n \in \mathbb{N}_0\}$ is a sequence of real-valued random variables adapted to that filtration. Suppose also that σ, σ_1, and σ_2 are stopping times for the filtration. For each of the following six cases either prove that τ is a stopping time or give an example to show that it need not be a stopping time.

(i) $\tau = \inf\{i \geq \sigma : X_i \in B\}$ for a given Borel set B

(ii) $\tau = \sigma_1 \wedge \sigma_2$ (minimum) or $\tau = \sigma_1 \vee \sigma_2$ (maximum)

(iii) $\tau = \sigma + 3$ or $\tau = (\sigma - 3)^+$

(iv) $\tau = \operatorname{argmax}_i\{X_i : 1 \leq i \leq k\}$.

*[2] Suppose $\Omega = (0, 1]$ and $\mathbb{P}$ equals Lebesgue measure restricted to $\mathcal{B}(0, 1]$. Suppose also that μ is another measure on $\mathcal{B}(0, 1]$ for which $\mu B \leq \mathbb{P}B$ for each $B \in \mathcal{B}(0, 1]$. Define $E_{i,k} = \big((i-1)/2^k, i/2^k\big]$ and $\mathcal{E}_k = \{E_{i,k} : 1 \leq i \leq 2^k\}$ and $\mathcal{E} = \cup_{k \in \mathbb{N}_0} \mathcal{E}_k$. Define $\mathcal{F}_n = \sigma(\mathcal{E}_n)$ and

$$X_k(\omega) = \sum\nolimits_{E \in \mathcal{E}_k} \{\omega \in E\} \frac{\mu E}{\mathbb{P}E}.$$

Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a martingale. Deduce that X_n converges both almost surely and in $\mathcal{L}^1$ to a random variable X for which $\mu B = \mathbb{P}(XB)$ for each $B \in \mathcal{B}(0, 1]$.

*[3] Suppose $S_1, \ldots, S_n$ is a nonnegative submartingale, with $\mathbb{P}S_i^p < \infty$ for some fixed $p > 1$. Let $q > 1$ be defined by $p^{-1} + q^{-1} = 1$. Show that $\mathbb{P}\left(\max_{i \leq n} S_i^p\right) \leq q^p \mathbb{P}S_n^p$, by following these steps.

(i) Write M_n for $\max_{i \leq n} S_i$. For fixed $x > 0$, and an appropriate stopping time τ, apply the Stopping Time Lemma to show that

$$x\mathbb{P}\{M_n \geq x\} \leq \mathbb{P}S_\tau\{S_\tau \geq x\} \leq \mathbb{P}S_n\{M_n \geq x\}.$$

(ii) Show that $\mathbb{P}X^p = \int_0^\infty px^{p-1}\mathbb{P}\{X \geq x\}\, dx$ for each nonnegative random variable X.

(iii) Show that $\mathbb{P}M_n^p \leq q\mathbb{P}S_n M_n^{p-1}$.

(iv) Bound the last product using Hölder's inequality, then rearrange to get the stated inequality. (Any problems with infinite values?)

[4] (HARDER) Suppose $\{Z_i : i \in \mathbb{N}_0\}$ is a sequence of random variables defined on Ω and $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$ for $n \in \mathbb{N}_0$. Let τ be a stopping time for that filtration. Remember that every $\mathcal{F}_n$-measurable random variable can be written in the form $g_n(Z_0, \ldots, Z_n)$ for some Borel measurable function g_n.

(i) Define $X_i = Z_{\tau \wedge i}$ and $\mathcal{G} := \sigma(X_i : i \in \mathbb{N}_0)$. Prove that X_i is $\mathcal{F}_\tau$-measurable. Deduce that $\mathcal{G} \subseteq \mathcal{F}_\tau$. Hint: Split $\{X_i \in B\}\{\tau \leq n\}$ into contributions from various sets $\{\tau = j\}$.

(ii) Prove that τ is $\mathcal{G}$-measurable. Hint: $\{\tau = 0\} = g_0(Z_0) = g_0(X_0)$ and $\{\tau = 1\} = g_1(Z_0, Z_1) = g_1(Z_0, Z_1)\{\tau \geq 1\} = g_1(X_0, X_1)\{\tau = 0\}^c$.

(iii) Show that $\mathcal{F}_\tau \subseteq \mathcal{G}$. Hint: If $F \in \mathcal{F}_\tau$ consider sets $F\{\tau = j\}$ for $j \in \mathbb{N}_0$.