Statistics 330b/600b, Math 330b spring 2017
Homework # 11
Due: Tuesday 25 April

*[1] Suppose:

(a) $\{(M_t, \mathcal{F}_t) : 0 \le t \le 1\}$ is a martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}M_t^2 < \infty$ for each t and $M_0 = 0$.

(b) There is a Markov kernel $\mathbb{K}$ from Ω to $[0,1]$ for which the process $A(t,\omega) = \mathbb{K}_\omega(0,t]$ is adapted to the filtration. That is, for each ω we have a probability measure $\mathbb{K}_\omega$ on $\mathcal{B}[0,1]$ for which the random variable $\mathbb{K}_\omega(0,t]$ is $\mathcal{F}_t$-measurable for each t. Write $\mathbb{Q}$ for the probability measure on $\mathcal{F} \otimes \mathcal{B}[0,1]$ defined by $\mathbb{Q}f(t,\omega) = \mathbb{P}^\omega \mathbb{K}_\omega^t f(t,\omega)$ for $f \in \mathcal{M}^+(\mathcal{F} \otimes \mathcal{B}[0,1])$.

(c) The process $Z_t = M_t^2 - A_t$ is a martingale for the same filtration.

(d) We are given a deterministic grid of points $0 = t_0 < t_1 < \cdots < t_{n+1} = 1$.

(i) Write $\Delta_i M$ for the increment $M(t_{i+1}, \omega) - M(t_i, \omega)$ for $i = 0, \dots, n$. Show that $\mathbb{P}_{\mathcal{F}_{t_i}}(\Delta_i M)^2 = \mathbb{P}_{\mathcal{F}_{t_i}} \mathbb{K}_\omega(t_i, t_{i+1}]$ a.e.$[\mathbb{P}]$ for each i.

(ii) Suppose we have a (predictable) process

$$H(t,\omega) = \sum_{i=0}^{n} h_i(\omega) \mathbb{1}\{t_i < t \le t_{i+1}\}$$

with $h_i \in \mathcal{L}^2(\omega, \mathcal{F}_{t_i})$ for each i. Define a new process $Y = H \bullet M$ by

$$Y(t,\omega) = \sum_{i=0}^{n} h_i(\omega) \big[M(t_{i+1} \wedge t, \omega) - M(t_i \wedge t, \omega)\big] \qquad \text{for } 0 \le t \le 1.$$

Show that $\mathbb{P}_{\mathcal{F}_t} Y_1 = Y_t$ a.e.$[\mathbb{P}]$. Deduce that $\{(Y_t, \mathcal{F}_t) : 0 \le t \le 1\}$ is a martingale. Hint: Suppose $t_j < t \le t_{j+1}$. Consider $\mathbb{P}_{\mathcal{F}_t}(h_i \Delta_i M)$ separately for each of the cases $i < j$ and $i = j$ and $i > j$.

(iii) Show that $\mathbb{P}Y_1^2 = \mathbb{Q}^{\omega,t} H(t,\omega)^2$.

*[2] Suppose P is a probability measure on the Borel sigma-field $\mathcal{B}(\mathcal{X})$ of a separable metric space $\mathcal{X}$. Suppose $(X_n : n \in \mathbb{N})$ is a sequence of random elements of $\mathcal{X}$ with the property that $\mathbb{P}\{X_n \in B\} \to PB$ for every P-continuity set. Let f be a bounded $\mathcal{B}(\mathcal{X})$-measurable function on $\mathcal{X}$ (with no loss of generality assume $0 \le f \le 1$) that is continuous at all points except those of a P-negligible set $\mathcal{N}$.

(i) For each real t, show that the boundary of the set $\{f \ge t\}$ is contained in $\mathcal{N} \cup \{f = t\}$. Deduce that $\{f \ge t\}$ is a P-continuity set for almost all (Lebesgue measure) t. Hint: Consider sequences $x_n \to x$ and $y_n \to x$ with $f(x_n) \ge t > f(y_n)$.

(ii) Deduce that $\mathbb{P}f(X_n) = \int_0^1 \mathbb{P}\{f(X_n) \ge t\}\, dt \to Pf$.

*[3] Suppose $(\mathcal{X}, d)$ is a metric space with a countable, dense subset $\{x_i : i \in \mathbb{N}\}$. Write $\mathcal{P}(\mathcal{X})$ for the set of all probability measures on $\mathcal{B}(\mathcal{X})$. For $P, Q \in \mathcal{P}(\mathcal{X})$ define

$$D(P, Q) = \sup\{|Pf - Qf| : \|f\|_{BL} \leq 1\}.$$

(i) Show that D is a metric on $\mathcal{P}(\mathcal{X})$.

(ii) For a given $\epsilon > 0$ define $h_0(x) \equiv \epsilon$ and $h_i(x) = (1 - d(x, x_i)/\epsilon)^+$ for $i \geq 1$. Define $H_k(x) = \sum_{i=0}^{k} h_i(x)$. Show that $\{H_k(x) \leq 1/2\} \downarrow \emptyset$ as $k \uparrow \infty$. Hint: What do you know about $H_k(x)$ if $k \geq i$ and $d(x, x_i) < \epsilon/2$?

(iii) Define $\ell_{i,k} = h_i/H_k$ for $0 \leq i \leq k$. Show that each $\ell_{i,k}$ belongs to $\text{BL}(\mathcal{X})$ and $\sum_{i=0}^{k} \ell_{i,k}(x) = 1$ for every x. Hint: First show that $1/H_k \in \text{BL}(\mathcal{X})$.

(iv) For each f with $\|f\|_{BL} \leq 1$ show that

$$|f(x) - \sum\nolimits_{i=1}^{k} f(x_i)\ell_{i,k}(x)| \leq \epsilon + f(x)\ell_{0,k}(x) \leq 3\epsilon + \{H_k(x) \leq 1/2\}.$$

(v) Suppose P and $\{P_n : n \in \mathbb{N}\}$ are probability measures on $\mathcal{B}(\mathcal{X})$ for which $P_n f \to Pf$ for each f in $\text{BL}(\mathcal{X})$. Show that $D(P_n, P) \to 0$. You may assume that $\limsup_n P_n F \leq PF$ for each closed set F.