Statistics 330b/600b, Math 330b spring 2017
Homework # 2
Due: Thursday 2 February

Please attempt at least the starred problems. Please explain your reasoning. Please, please, please look at the handout latex.pdf for an explanation of why solutions consisting of a long string of sequence of $\therefore$'s and $\because$'s can be hard to follow.

*[1] Suppose $(\mathcal{X}, \mathcal{A}, \mu)$ is a measure space with $\mu\mathcal{X} < \infty$. Suppose also that $\{f_n : n \in \mathbb{N}\}$ is a sequence of $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable real functions for which $f_n(x) \to 0$ as $n \to \infty$, for each $x \in \mathcal{X}$. Show that $\mu\{x \in \mathcal{X} : |f_n(x)| \geq \epsilon\} \to 0$ as $n \to \infty$ for each fixed $\epsilon > 0$. Hint: Think Dominated Convergence with $g_n(x) = \mathbb{1}\{|f_n| \geq \epsilon\}$.

*[2] Suppose $\{f_n : n \in \mathbb{N}\}$ is a sequence of real-valued $\mathcal{B}(\mathcal{R})$-measurable functions on a measure space $(\mathcal{X}, \mathcal{A}, \mu)$ for which: for each $\epsilon > 0$ and $\delta > 0$ there exists an $n_{\epsilon,\delta}$ for which

$$\mu\{x : |f_n(x) - f_m(x)| > \delta\} < \epsilon \qquad \text{whenever } m, n \geq n_{\epsilon,\delta}.$$

(i) Write $n(k)$ for the $n_{\epsilon,\delta}$ corresponding to $\epsilon = \delta = 2^{-k}$, for $k \in \mathbb{N}$. [You may assume that $n(1) < n(2) < \dots$. Why?] Show that

$$\mu \sum\nolimits_{k\in\mathbb{N}} \mathbb{1}\{x : |f_{n(k)}(x) - f_{n(k+1)}(x)| > 2^{-k}\} < \infty.$$

(ii) Deduce that there exists a μ-negligible set N for which $\{f_{n(k)}(x) : k \in \mathbb{N}\}$ is a Cauchy sequence of real numbers for each x in N^c. Define

$$f(x) := \limsup\nolimits_k f_{n(k)}(x)\mathbb{1}\{x \in N^c\}.$$

Show that $f_{n(k)}(x) \to f(x)$ as $k \to \infty$, for each x in N^c.

*[3] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose $\{X_i : i \in \mathbb{N}\}$ are real-valued random variables (that is, $\mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurable real-valued functions) all defined on Ω. Let $\mathcal{B}$ denote the product sigma-field on $\mathbb{R}^{\mathbb{N}}$.

(i) Show that the map $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ defined by $X(\omega) = (X_1(\omega), X_2(\omega), \dots)$ is $\mathcal{F}\backslash\mathcal{B}$-measurable. Write P for the distribution of X.

(ii) Define $S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $Sx = y$ where $y_i = x_{i+1}$ for $i \in \mathbb{N}$. Show that S is $\mathcal{B}\backslash\mathcal{B}$-measurable. Write Q for the image of P under S.

(iii) Suppose that, for each m and k in $\mathbb{N}$, the random vector $(X_1, \dots, X_m)$ has the same distribution as the random vector $(X_{1+k}, \dots, X_{m+k})$. Show that $Q = P$. Hint: Define $\mathcal{C}$ as the collection of all sets of the form

$$\{x \in \mathbb{R}^{\mathbb{N}} : (x_1, \dots, x_m) \in D\}$$

where m ranges over $\mathbb{N}$ and D ranges over all product-measurable subsets of $\mathbb{R}^m$. Show that $\mathcal{C}$ is a field for which $\mathcal{B} = \sigma(\mathcal{C})$ then use a generating class argument.

$\boxed{\text{PTO}}$

[4] For each i in some index set I suppose $\mathcal{X}_i$ is a set equipped with a sigma-field $\mathcal{A}_i$. The product space $\mathcal{X}_I = \prod_{i \in I} \mathcal{X}_i$ is defined as the set of all functions $x : I \to \cup_{i \in I} \mathcal{X}_i$ such that $x_i \in \mathcal{X}_i$ for each i. Let $\mathcal{E}_i$ denote the collection of all sets of the form $\{x \in \mathcal{X}_I : x_i \in A\}$ with $A \in \mathcal{A}_i$. For each $S \subseteq I$ define $\mathcal{A}_S = \sigma\left(\cup_{i \in S} \mathcal{E}_i\right)$.

(i) Suppose I is uncountable. Show that $\mathcal{A}_I = \cup\{\mathcal{A}_S : \ S \text{ is a countable subset of } I\}$.

(ii) Again suppose I is uncountable. For each f in $\mathcal{M}^+(\mathcal{X}_I, \mathcal{A}_I)$ show that there exists a countable subset S of I such that $f(x) = g_S(x_S)$, where x_S denotes the restriction of x to S and g_S is a suitably measurable function on $\mathcal{X}_S$.