Statistics 330b/600b, Math 330b spring 2017
Homework # 3
Due: Thursday 9 February

Please attempt at least the starred problems.

*[1] Suppose $(\mathcal{X}, \mathcal{A}, \mu)$ is a measure space with $\mu\mathcal{X} < \infty$. Suppose also that $g_1, g_2 \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ have the property that $\mu(g_1 \mathbb{1}_A) = \mu(g_2 \mathbb{1}_A)$ for all A in $\mathcal{A}$. Prove that $g_1 = g_2$ a.e.$[\mu]$. Hint: Consider sets $A_{r,s} = \{x \in \mathcal{X} : g_1(x) < r < s < g_2(x)\}$. What do you know about $\mu\left(g_i \mathbb{1}_{A_{r,s}}\right)$?

*[2] Suppose $\{f_n : n \in \mathbb{N}\} \subset \mathcal{L}^\Psi(\mathcal{X}, \mathcal{A}, \mu)$ where $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is convex and nondecreasing, with $\Psi(0) = 0$ and $\Psi(r) \to \infty$ as $r \to \infty$. Suppose the sequence is Cauchy: for each $\epsilon > 0$ there exists an n_δ such that $\|f_n - f_m\|_\Psi < \delta$ for $m, n \geq n_\delta$.

(i) Choose r so large that $\Psi(r) > 1/\epsilon$. Show that, for $\min(n, m) > n_{\delta/r}$,

$$\mu\{x : |f_n(x) - f_m(x)| > \delta\} \leq \mu\Psi\left(r|f_n(x) - f_m(x)|/\delta\right)/\Psi(r) < \epsilon$$

(ii) Deduce from HW2.2 that there exists an $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable real-valued function f and a subsequence along which $f_{n(k)}(x) \to f(x)$ a.e.$[\mu]$. If n is large enough invoke Fatou to show that

$$1 \geq \liminf_{k\to\infty} \mu\Psi\left(|f_n(x) - f_{n(k)}(x)|/\epsilon\right) \geq \mu\Psi\left(|f_n(x) - f(x)|/\epsilon\right).$$

(iii) Deduce that $\|f_n - f\|_\Psi \to 0$.

*[3] Suppose $f_1, \ldots, f_k \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\theta_1, \ldots, \theta_k$ are strictly positive numbers that sum to one. Let μ be a measure on $\mathcal{A}$. Show that

$$\mu \prod\nolimits_{i\leq k} f_i^{\theta_i} \leq \prod\nolimits_{i\leq k} (\mu f_i)^{\theta_i}$$

by following these steps. You may use the inequality

<1> $$\sum\nolimits_{i=1}^k a_i^{\theta_i} \leq \sum\nolimits_{i=1}^k \theta_i a_i \qquad \text{for all } a_i \in [0, \infty),$$

which, as explained in class, is a simple consequence of the concavity of the log function.

(i) Explain why the inequality is trivially true if $\mu f_i = 0$ for at least one i.

(ii) Explain why the inequality is trivially true if $\mu f_i > 0$ for all i and $\mu f_i = +\infty$ for at least one i.

(iii) Explain why there is no loss of generality in assuming that $\mu f_i = 1$ for each i and $f_i(x) < \infty$ for each x and i.

(iv) Complete the proof by considering the inequality <1> with $a_i = f_i(x)$.

Remark. Textbooks often contain the the special case where $k = 2$ and $\theta_1 = 1/p$ and $\theta_2 = 1/q$ and $f_1 = |g_1|^p$ and $f_2 = |g_2|^q$, with the assertion that $|\mu(g_1 g_2)| \leq \mu|g_1 g_2| \leq \left(\mu|g_1|^p\right)^{1/p}\left(\mu|g_2|^q\right)^{1/q}$.

[4] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and μ is a measure on $\mathcal{A}$. Write $\mathcal{N}_\mu$ for $\{N \in \mathcal{A} : \mu N = 0\}$. Define

$$\mathcal{A}_\mu := \{B \subseteq \mathcal{X} : \exists A \in \mathcal{A},\ N \in \mathcal{N}_\mu \text{ such that } |\mathbb{1}_B - \mathbb{1}_A| \le \mathbb{1}_N\}.$$

(i) Show that $\mathcal{A}_\mu$ is a sigma-field.

(ii) If $|\mathbb{1}_B - \mathbb{1}_{A_i}| \le \mathbb{1}_{N_i}$ for $i = 1, 2$, with $A_i \in \mathcal{A}$ and $N_i \in \mathcal{N}_\mu$, show that $\mu A_1 = \mu A_2$.

(iii) If $|\mathbb{1}_B - \mathbb{1}_A| \le \mathbb{1}_N$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}_\mu$ define $\nu B = \mu A$. Show that ν is a well defined measure on $\mathcal{A}_\mu$ whose restriction to $\mathcal{A}$ equals μ.