Statistics 330b/600b, Math 330b spring 2017
Homework # 4
Due: Thursday 16 February

Please attempt at least the starred problems.

*[1] Let $A_1, A_2, \ldots$ be events (members of $\mathcal{F}$) in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $S_n = A_1 + \cdots + A_n$ and $S_\infty = \sum_{i\in\mathbb{N}} A_i$ (sums of indicator functions). Suppose $\|S_n\|_1 \to \infty$ and $\|S_n\|_2 / \|S_n\|_1 \to 1$ as $n \to \infty$. In class I showed that these assumptions imply $\mathbb{P}\{S_\infty \geq 1\} = 1$.

(i) For a fixed $m \in \mathbb{N}$ define $T_n = S_n - S_m$ for $n > m$. Show that$\|T_n\| \to \infty$ and $\|T_n\|_2 / \|T_n\|_1 \to 1$ as $n \to \infty$. Deduce that

$$\mathbb{P}\{S_\infty - S_m \geq 1\} = 1 \qquad \text{for each } m \in \mathbb{N}.$$

Hint: Bound $\|S_n - T_n\|$ for both $\mathcal{L}^1$ and $\mathcal{L}^2$ norms.

(ii) Deduce that $\mathbb{P}\{\omega \in A_i$ for infinitely many i $\} = 1$.

(iii) Suppose $\{B_i : i \in \mathbb{N}\}$ is a sequence of events for which $\sum_{i\in\mathbb{N}} \mathbb{P}B_i = \infty$ and $\mathbb{P}(B_iB_j) = (\mathbb{P}B_i)(\mathbb{P}B_j)$ for all $i \neq j$. Show that

$$\mathbb{P}\{\omega \in B_i \text{ for infinitely many } i\ \} = 1.$$

Please use (ii). I am not interested in seeing the standard textbook proof for the harder direction of Borel-Cantelli.

*[2] Suppose $\Omega = \{1, 2, 3, 4\}$ and $\mathbb{P}$ is the probability measure that puts mass 1/4 on each point of Ω. Define $\mathcal{E}_1 = \{\Omega, \{1, 2\}\}$ and $\mathcal{E}_2 = \{\Omega, \{2, 3\}, \{2, 4\}\}$.

(i) Show that $\mathbb{P}(E_1E_2) = (\mathbb{P}E_1)(\mathbb{P}E_2)$ for all choices of $E_i \in \mathcal{E}_i$, for $i = 1, 2$.

(ii) Show that $\sigma(\mathcal{E}_1)$ and $\sigma(\mathcal{E}_2)$ are not independent. Why does this fact not contradict the result proved in class?

*[3] Suppose $\mathcal{X}_i$ is a topological space with $\mathcal{G}_i$ the collection of all its open subsets, for $i = 1, 2$. By definition, the Borel sigma-field $\mathcal{B}(\mathcal{X}_i)$ equals $\sigma(\mathcal{G}_i)$. The product topology $\mathcal{G}$ on $\mathcal{X}_1 \times \mathcal{X}_2$ is defined as the collection of sets expressible as unions of (possibly uncountably many) sets of the form $G_1 \times G_2$, with $G_i \in \mathcal{G}_i$. The Borel sigma-field $\mathcal{B}(\mathcal{X}_1 \times \mathcal{X}_2)$ is defined to be $\sigma(\mathcal{G})$.

(i) Define $\mathcal{F}_1 = \{B \in \mathcal{B}(\mathcal{X}_1) : B \times \mathcal{X}_2 \in \mathcal{B}(\mathcal{X}_1 \times \mathcal{X}_2)\}$. Use a generating class argument to show that $\mathcal{F}_1 = \mathcal{B}(\mathcal{X}_1)$.

(ii) Show that $B_1 \times B_2 \in \mathcal{B}(\mathcal{X}_1 \times \mathcal{X}_2)$ for all $B_i \in \mathcal{B}(\mathcal{X}_i)$. Deduce that $\mathcal{B}(\mathcal{X}_1 \times \mathcal{X}_2) \supseteq \mathcal{B}(\mathcal{X}_1) \otimes \mathcal{B}(\mathcal{X}_2)$, the product sigma-field on $\mathcal{X}_1 \times \mathcal{X}_2$.

(iii) Say that $\mathcal{G}_i$ is countably generated if it has a countable subcollection $\mathcal{H}_i$ such that every G in $\mathcal{G}_i$ is a union of sets from $\mathcal{H}_i$. Show that

$$\mathcal{B}(\mathcal{X}_1 \times \mathcal{X}_2) = \mathcal{B}(\mathcal{X}_1) \otimes \mathcal{B}(\mathcal{X}_2)$$

if both $\mathcal{G}_1$ and $\mathcal{G}_2$ are countably generated.

[4] Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\mathcal{F}_i : i \in I\}$ is a (possibly infinite) collection of sub-sigma-fields that is independent. Suppose I_1 and I_2 are disjoint (non-empty) subsets of I. Define $\mathcal{G}_j = \sigma\left(\cup_{i\in I_j} \mathcal{F}_i\right)$. Show that $\mathcal{G}_1$ and $\mathcal{G}_2$ are independent.

[5] Let $\Psi(x) = e^{x^2} - 1$ for $x \geq 0$. Suppose $W = (W_1, \dots, W_k)$ is a random vector on $(\Omega, \mathcal{F}, \mathbb{P})$ for which $\|\langle W, u\rangle\|_\Psi \leq 1$ for every u in the set of all unit vectors in $\mathbb{R}^k$. Remember that $|W| = \sqrt{\sum_i W_i^2} = \sup\{\langle W, u\rangle : u \in \mathcal{U}\}$.

(i) Let $\mathcal{U}_0$ be a maximal subset of $\mathcal{U}$ for which $|u - v| > 1/2$ for each distinct pair u, v in $\mathcal{U}_0$. Show that $\mathcal{U}_0$ has cardinality at most 5^k. Hint: The balls of radius $1/4$ centered at the points of $\mathcal{U}_0$ are disjoint and their union lies in a ball of radius $5/4$.

(ii) Explain why for each $v \in \mathcal{U}$ there exists a $u \in \mathcal{U}_0$ such that $|u - v| \leq 1/2$.

(iii) Deduce that

$$\langle W, u\rangle \leq \max_{u \in \mathcal{U}_0} \langle W, u\rangle + \tfrac{1}{2}|W|$$

for every v in $\mathcal{U}$.

(iv) Deduce that there exists a universal constant C for which $\mathbb{P}|W| \leq C\sqrt{k}$, for every $k \in \mathbb{N}$.