Statistics 330b/600b, Math 330b spring 2017
Homework # 6
Due: Thursday 2 March

*[1] Suppose μ and ν are sigma-finite measures on $\mathcal{B}(\mathbb{R})$ and $\lambda = \mu \otimes \nu$. Define $\mathbb{A}_\mu = \{x \in \mathbb{R} : \mu\{x\} > 0\}$ and $\mathbb{A}_\nu = \{y \in \mathbb{R} : \nu\{y\} > 0\}$.

(i) Explain why the set $D := \{(x, y) \in \mathbb{R}^2 : x = y\}$ (the 'diagonal') belongs to $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

(ii) Show that the set $\mathbb{A}_\mu$ of atoms is at worst countably infinite. Hint: How many points x with $\mu\{x\} \geq 1/n$ can there be inside a set B with $\mu B < \infty$?

(iii) Explain why the function $h(x) := \nu^y\{x = y\}$ is well defined and $\mathcal{B}(\mathcal{R})$-measurable. Show that $h(x) = \sum_{\alpha \in \mathbb{A}_\mu} \nu\{\alpha\}\mathbb{1}\{x = \alpha\}$.

(iv) Show that $\lambda(D) = \sum_{\alpha \in \mathbb{A}_\mu \cap \mathbb{A}_\nu} \mu\{\alpha\}\nu\{\alpha\}$.

*[2] Suppose $f \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ and $g \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, with μ and ν as in Problem [1].

(i) Show that the function $\psi(x, y) = f(x)g(y)$ belongs to $\mathcal{L}^1(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda)$.

(ii) Define $F(y) := \mu^x \left(f(x)\{x \leq y\}\right)$ and $G(x) := \nu^y \left(g(y)\{y \leq x\}\right)$. Show that

$$\mu^x \left(G(x)f(x)\right) + \nu^y \left(F(y)g(y)\right) = (\mu f)(\nu g) + \sum_{\alpha \in \mathbb{A}_\mu \cap \mathbb{A}_\nu} f(\alpha)g(\alpha)\mu\{\alpha\}\nu\{\alpha\}.$$

Hint: $\mathbb{1}\{(x, y) \in \mathbb{R}^2 : x \leq y\} + \mathbb{1}\{(x, y) \in \mathbb{R}^2 : x \geq y\} =?$.

Remark. This result generalizes the classical formula for integration by parts.

*[3] Define $\Psi(x) = e^{x^2} - 1$ for $x \geq 0$. Let X be a random variable, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that $X \in \mathcal{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$ if and only if, for some positive constants c_1, c_2, and c_3,

$$\mathbb{P}\{|X| \geq x\} \leq c_1 \exp(-c_2 x^2) \qquad \text{for all } |x| \geq c_3.$$

Hint: First show that $\Psi(x) = \int_0^\infty e^t \mathbb{1}\{t \leq x^2\}\, dt$.

*[4] Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be two measure spaces, with both μ and ν sigma-finite. Write $\mathcal{G}$ for the set of all functions expressible as finite linear combinations of indicators of measurable rectangles. That is, a typical g in $\mathcal{G}$ is expressible as a finite sum $\sum_{i=1}^k \alpha_i \mathbb{1}\{x \in A_i,\, y \in B_i\}$ for some sets $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$ and real numbers α_i, for $i = 1, 2, \ldots, k$.

Show that for each f in $\mathcal{L}^1(\mathcal{X} \times \mathcal{Y}, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ and each $\epsilon > 0$ there exist a $g \in \mathcal{G}$ such that $\mu \otimes \nu|f - g| < \epsilon$. Follow these steps.

(i) First suppose that both μ and ν are finite measures and $|f|$ is bounded. Use a lambda-space argument to establish the asserted approximation property.

(ii) Extend to the sigma-finite case.

[5] Suppose $(\mathcal{X}, \mathcal{A}, \mu)$ is a sigma-finite measure space. The weak $\mathcal{L}^q(\mu)$ "norm" of a measurable function f is defined as $\|f\|_{q,\infty} := \sup_{t>0} t\mu\{|f| > t\}^{1/q}$.

(i) Show that $\|f\|_{q,\infty} \leq \|f\|_q := (\mu|f|^q)^{1/q}$.

(ii) Show $\|\lambda f\|_{q,\infty} = \lambda \|f\|_{q,\infty}$ for each positive constant λ.

(iii) Suppose $1 \leq r < q < \infty$. Show that $\|f\|_{q,\infty} \geq b_{r,q} \|f\|_r$ where $b_{r,q}$ denotes the constant $(1 - (r/q))^{1/r}$.

(iv) Does $\|\cdot\|_{q,\infty}$ satisfy the triangle inequality?