Statistics 330b/600b, Math 330b spring 2017
Homework # 8
Due: Thursday 30 March

Radon-Nikodym

[1] Suppose λ and ν are both finite measures both defined on $(\mathcal{X}, \mathcal{A})$. Suppose also that ν is dominated by μ: if $A \in \mathcal{A}$ and $\lambda A = 0$ then $\nu A = 0$. Follow these steps to show that there exists a real-valued function in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ for which

$$\nu f = \lambda(f\Delta) \qquad \text{for each } f \in \mathcal{M}^+(\mathcal{X}, \mathcal{A}).$$

(i) Define $\mu = \lambda + \nu$. Deduce from the projection.pdf handout that there exists an $\mathcal{A}$-measurable function Δ_0, with $0 \le \Delta_0(x) \le 1$ for all x, such that $\nu f = \mu(f\Delta_0)$ for each $f \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$. (No need to repeat the whole proof from the handout.)

(ii) Show that $\mu\{\Delta_0 = 1\} = 0$. Hint: $\nu\{\Delta_0 = 1\} = ??$.

(iii) Define $\Delta = \{\Delta_0 < 1\}\Delta_0/(1 - \Delta_0)$. For a given f in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and each i in $\mathbb{N}$ define

$$f_i = \left(\frac{f \wedge i}{1 - \Delta_0}\right)\{\Delta_0 \le 1 - i^{-1}\}$$

Rearrange terms in the equality $\nu f_i = \mu(f_i\Delta_0)$, explaining why there are no $\infty - \infty$ problems, then let i tend to infinity.

(iv) Extend the result to the case of sigma-finite measures λ and ν under the same domination condition.

Conditioning

The projection.pdf handout (Section 4) described the traditional approach to Kolmogorov conditional expectations where everything is carried out on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the conditioning information is given by a sub-sigma-field $\mathcal{G}$ of $\mathcal{F}$. The handout described the conditional expectation as a map $\mathbb{P}_\mathcal{G}$ from $\mathcal{M}^+(\Omega, \mathcal{F})$ into $\mathcal{M}^+(\Omega, \mathcal{G})$, with $\mathbb{P}_\mathcal{G} f$ defined only up to a $\mathbb{P}$-equivalence, having the properties:

(a) $\mathbb{P}_\mathcal{G} 0 = 0$ *and* $\mathbb{P}_\mathcal{G} 1 = 1$ *a.e.*$[\mathbb{P}]$;

(b) $\mathbb{P}_\mathcal{G}(c_1 Y_1 + c_2 Y_2) = c_1 \mathbb{P}_\mathcal{G} Y_1 + c_2 \mathbb{P}_\mathcal{G} Y_2$ *a.e.*$[\mathbb{P}]$ *for constants* $c_i \in \mathbb{R}^+$;

(c) $\mathbb{P}_\mathcal{G} Y_1 \le \mathbb{P}_\mathcal{G} Y_2$ *a.e.*$[\mathbb{P}]$ *if* $Y_1(\omega) \le Y_2(\omega)$ *for all* ω;

(d) if $Y_n(\omega) \uparrow Y(\omega)$ *then* $\mathbb{P}_\mathcal{G} Y_n \uparrow \mathbb{P}_\mathcal{G} Y$ *a.e.*$[\mathbb{P}]$;

(e) if $G \in \mathcal{M}^+(\omega, \mathcal{G})$ *and* $Y \in \mathcal{M}^+(\omega, \mathcal{F})$ *then* $\mathbb{P}_\mathcal{G}(GY) = G\mathbb{P}_\mathcal{G} Y$ *a.e.*$[\mathbb{P}]$;

(f) $\mathbb{P} Y = \mathbb{P}(\mathbb{P}_\mathcal{G} Y)$ *a.e.*$[\mathbb{P}]$ *for each* $Y \in \mathcal{M}^+(\omega, \mathcal{F})$.

It also mentioned that $\mathbb{P}_\mathcal{G}$ *has a modification as a map from* $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ *to* $\mathcal{L}^1(\Omega, \mathcal{G}, \mathbb{P})$.

Problem [2] presents a more direct way to get the second form of $\mathbb{P}_\mathcal{G}$ *as an extension of projection from domain* $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ *to domain* $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. *To stress the analogy with Markov kernels I write* $\mathcal{K}_\omega f$ *instead of* $\mathbb{P}_\mathcal{G} f$.

For Problem [2] I also ask you to prove the conditional form of Dominated Convergence, rather than the conditional form of Monotone Convergence. MC seems more appropriate with $\mathcal{M}^+(\mathcal{F})$, *where we do not need to worry about finiteness or integrability for limits.*

[2] Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$. Abbreviate $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathcal{L}^2(\mathcal{F})$ and $\mathcal{L}^2(\Omega, \mathcal{G}, \mathbb{P})$ to $\mathcal{L}^2(\mathcal{G})$. Abbreviate $\mathcal{M}_{\text{bdd}}(\Omega, \mathcal{G})$ to $\mathcal{M}_{\text{bdd}}(\mathcal{G})$.

(i) Show that $\mathcal{L}^2(\mathcal{G})$ is a closed subspace of $\mathcal{L}^2(\mathcal{F})$, in the sense defined on the projection.pdf handout.

(ii) For each $f \in \mathcal{L}^2(\mathcal{F})$ write $\pi_\omega f$ for a function (chosen arbitrarily from the equivalence class of possibilities) in $\mathcal{L}^2(\mathcal{G})$ for which $f - \pi_\omega f \perp \mathcal{L}^2(\mathcal{G})$. For all f, f_1, f_2 in $\mathcal{L}^2(\mathcal{F})$ show that

- (a) $\pi_\omega 0 = 0$ and $\pi_\omega 1 = 1$ a.e.[$\mathbb{P}$]
- (b) $\mathbb{P}\pi_\omega f = \mathbb{P}f$
- (c) $\pi_\omega(G_1 f_1 + G_2 f_2) = G_1(\omega)\pi_\omega f_1 + G_2(\omega)\pi_\omega f_2$ a.e.[$\mathbb{P}$] for all $G_1, G_2 \in \mathcal{M}_{\text{bdd}}(\mathcal{G})$
- (d) if $f \geq 0$ a.e.[$\mathbb{P}$] then $\pi_\omega f \geq 0$ a.e.[$\mathbb{P}$]

(iii) Suppose $f \in \mathcal{L}^1(\mathcal{F})$ and $\{f_n : n \in \mathbb{N}\}$ is a sequence in $\mathcal{L}^2(\mathcal{F})$ for which $\mathbb{P}|f_n - f| \to 0$ as $n \to \infty$. Use (c) and (d) to show that $\{\pi_\omega f_n : n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{L}^1(\mathcal{G})$. Deduce that there is a $g(\omega) \in \mathcal{L}^1(\mathcal{G})$ for which $\mathbb{P}|\pi_\omega f_n - g| \to 0$. Hint: First use (d) to show that $|\pi_\omega h| \leq \pi_\omega |h|$ a.e.[$\mathbb{P}$] for each h in $\mathcal{L}^2(\mathcal{F})$.

(iv) With f_n, f, g as in part (iii), show that

<1> $$\mathbb{P}fG = \mathbb{P}gG \qquad \text{for each } G \in \mathcal{M}_{\text{bdd}}(\mathcal{G}).$$

Also show that if g_1 is another function in $\mathcal{L}^1(\mathcal{G})$ that satisfies an analogous set of equalities then $g_1 = g$ a.e.[$\mathbb{P}$]. *(Hint: You solved a similar problem on HW3.)* Denote by $\mathcal{K}_\omega f$ any g in $\mathcal{L}^1(\mathcal{G})$ (chosen arbitrarily from the $\mathbb{P}$-equivalence class) for which <1> holds.

(v) For all $f, f_1, f_2 \in \mathcal{L}^1(\mathcal{F})$ prove that the analogs of the four properties listed in (ii) hold if π_ω is replaced by $\mathcal{K}_\omega$. *Hint: You could approximate f, f_1, f_2 in the $\mathcal{L}^1(\mathcal{F})$ sense by functions from $\mathcal{L}^2(\mathcal{F})$, as in part (iii), then deduce the results as limiting forms of the corresponding results from (ii). Alternatively, you could argue directly from <1>, using (iii) purely as an existence proof. For example, if $g_i = \mathcal{K}_\omega f_i$ then you should explain why $\mathbb{P}(f_1 + f_2 - g_1 - g_2)G = 0$ for each $G \in \mathcal{M}_{\text{bdd}}(\mathcal{G})$.*

(vi) Suppose $\{f_n : n \in \mathbb{N}\} \subset \mathcal{L}^1(\mathcal{F})$ and $f_n(\omega) \to f(\omega)$ for each ω (or even just a.e.[$\mathbb{P}$]). Suppose also that there is an F in $\mathcal{L}^1(\mathcal{F})$ for which $\sup_n |f_n(\omega)| \leq F(\omega)$ for every ω. Show that $\mathcal{K}_\omega f_n \to \mathcal{K}_\omega f$ a.e.[$\mathbb{P}$]. Hint: Show that $2F(\omega) \geq F_n(\omega) := \sup_{i \geq n} |f_n(\omega) - f(\omega)| \downarrow 0$ and $\mathbb{P}F_n \downarrow 0$ and $\mathcal{K}_\omega F_n(\omega) \geq \sup_{i \geq n} |\mathcal{K}_\omega f_n - \mathcal{K}_\omega f|$ a.e.[$\mathbb{P}$].

[3] Here is an alternative to Problem [2], which shows how to extend the projection map π on $\mathcal{L}^2(\mathcal{F})$ to a map on $\mathcal{M}^+(\mathcal{F})$ with the properties for $\mathbb{P}_\mathcal{G}$ listed on the previous page. For each f in $\mathcal{M}^+(\mathcal{F})$ define

$$\mathcal{K}_\omega f := g(\omega) := \sup\nolimits_{i \in \mathbb{N}} \pi_\omega f_i \qquad \text{where } f_i = f \wedge i.$$

(i) Explain why $g_i(\omega) := \pi_\omega f_i \uparrow g(\omega) \in \mathcal{M}^+(\mathcal{G})$ a.e.[$\mathbb{P}$].

(ii) Explain why

<2> $$\mathbb{P}(fG) = \lim\nolimits_i \mathbb{P}(f_i G) = \lim\nolimits_i \mathbb{P}(g_i G) = \mathbb{P}(gG).$$

for each G in the set $\mathcal{M}^+_{\text{bdd}}(\mathcal{G})$ of all bounded, nonnegative, $\mathcal{G}$-measurable functions.

(iii) Show that equality <2> characterizes $g \in \mathcal{M}^+(\mathcal{G})$ up to $\mathbb{P}$-equivalence.

(iv) Use equality <2> to establish the properties for $\mathbb{P}_\mathcal{G}$ listed on the previous page. Note that you cannot be as free with subtraction as with part (v) of Problem [2]. For example, if $f_i \in \mathcal{M}^+(\mathcal{F})$ and $g_i(\omega) = \mathcal{K}_\omega f_i$ then why is the equality $\mathbb{P}(f_1 + f_2)G = \mathbb{P}(g_1 + g_2)G$ true but $\mathbb{P}(f_1 + f_2 - g_1 - g_2)G = 0$ is suspect?