Statistics 330b/600b, Math 330b spring 2017
Homework # 9
Due: Thursday 6 April

*[1] Let $\mathcal{P} = \{\mathbb{P}_\theta : \theta \in \Theta\}$ be a set of probability measures all defined on the same $(\Omega, \mathcal{F})$. Suppose also that there exists a sigma-finite measure λ on $\mathcal{F}$ that dominates each $\mathbb{P}_\theta$, with $d\mathbb{P}_\theta/d\lambda = p(\omega, \theta) = p_\theta(\omega)$.

(i) Suppose $\{\Omega_i : i \in \mathbb{N}\}$ is a partition of Ω into $\mathcal{F}$-measurable sets with $0 < \lambda\Omega_i < \infty$ for each i. Define μ as the measure with density $\gamma(\omega) = \sum_{i\in\mathbb{N}} 2^{-i}\mathbb{1}\{\omega \in \Omega_i\}/\lambda\Omega_i$ with respect to λ. Show that $\mu\Omega < \infty$ and $d\mathbb{P}_\theta/d\mu = p(\omega, \theta)/\gamma(\omega)$.

(ii) Define $G_\theta = \{\omega \in \Omega : p_\theta(\omega) > 0\}$. Write $\mathcal{D}$ for the collection of all countable subsets of Θ. For each D in $\mathcal{D}$ define $G_D = \cup_{\theta\in D} G_\theta$. Show that there exists an S in $\mathcal{D}$ for which $\mu G_S = \sup_{D\in\mathcal{D}} \mu G_D$.

(iii) Enumerate the S from part (ii) as $\{\theta_i : i \in \mathbb{N}\}$. Define $\mathbb{P} = \sum_{i\in\mathbb{N}} 2^{-i}\mathbb{P}_{\theta_i}$. Show that $d\mathbb{P}/d\mu = \rho(\omega)/\gamma(\omega)$ where $\rho(\omega) := \sum_{i\in\mathbb{N}} 2^{-i}p(\omega, \theta_i)$.

(iv) For each θ, show that $\mathbb{P}_\theta$ is dominated by $\mathbb{P}$ with density

$$d\mathbb{P}_\theta/d\mathbb{P} = \mathbb{1}\{\omega : \rho(\omega) > 0\}p(\omega, \theta)/\rho(\omega).$$

Hint: First show that $G_S = \{\rho > 0\}$ and $\mu\,(G_\theta \backslash G_S) = 0$.

*[2] The handout ***Ergodic.pdf*** explained what it means for a measurable map $T : \Omega \to \Omega$ to be measure preserving, for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It also defined an $\mathcal{F}$-measurable real-valued function h on Ω to be ***invariant*** if $h(T\omega) = h(\omega)$ for each ω.

Suppose g is an $\mathcal{F}$-measurable real-valued function on Ω for which $g(T\omega) = g(\omega)$ a.e.[$\mathbb{P}$]. Show that there is an invariant h for which $g(\omega) = h(\omega)$ a.e.[$\mathbb{P}$]. *Hint: Consider* $\limsup n^{-1} \sum_{0\le i<n} g(T^i\omega)/n$.

*[3] The handout ***Ergodic.pdf*** proved that if $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and

$$S_n(\omega) = f(\omega) + f(T\omega) + \cdots + f(T^{n-1}\omega)$$

for a measure preserving transformation T then

$$S_n(\omega)/n \to Z(\omega) = \mathbb{P}_\mathcal{J} f \qquad \text{a.e.}[\mathbb{P}].$$

The handout also asserted that $\mathbb{P}|S_n/n - Z| \to 0$. Prove this assertion.
Hint: For an appropriate constants C, split f into a sum of $f_1 = f\{|f| \le C\}$ and $f_2 = f\{|f| > C\}$, with corresponding decompositions $S_n = S_{n,1} + S_{n,2}$ and $\mathbb{P}_\mathcal{J} f = Z_1 + Z_2$.