Statistics 330/600, springl 2017
Homework #10 solutions

*[1] *Suppose $\{\mathcal{F}_n : n \in \mathbb{N}_0\}$ is a filtration defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{X_n : n \in \mathbb{N}_0\}$ is a sequence of real-valued random variables adapted to that filtration. Suppose also that σ, σ_1, and σ_2 are stopping times for the filtration. For each of the following six cases either prove that τ is a stopping time or give an example to show that it need not be a stopping time.*

(i) $\tau = \inf\{i \geq \sigma : X_i \in B\}$ *for a given Borel set B*

SOLUTION: *For each $n \in \mathbb{N}_0$,*

$$\{\tau(\omega) \leq n\} = \cup_{j=0}^{n} \cup_{i=j}^{n} \{\sigma(\omega) = j,\ X_i(\omega) \in B\}.$$

Check that $\{\sigma = j,\ X_i \in B\} \in \mathcal{F}_i \subseteq \mathcal{F}_n$.

(ii) $\tau = \sigma_1 \wedge \sigma_2$ *(minimum) or* $\tau = \sigma_1 \vee \sigma_2$ *(maximum)*

(iii) $\tau = \sigma + 3$ *or* $\tau = (\sigma - 3)^+$

(iv) $\tau = \operatorname{argmax}_i\{X_i : 1 \leq i \leq k\}$.

SOLUTION: *Interpret the argmax as the first i for which $X_i(\omega) = M(\omega) = \max_{1\leq j\leq k} X_j(\omega)$. Then*

$$\{\omega : \tau(\omega) \leq 1\} = \{\omega : X_1(\omega) = M(\omega)\} = \{\omega : X_1(\omega) \geq \max_{2\leq j\leq k} X_j(\omega)\}.$$

You needed to create an example where this set is not $\mathcal{F}_1$-measurable. For example, suppose $\Omega = \{-1, 1\}^k$ with $\xi_i(\omega)$ being the ith coordinate and $\mathcal{F}_i = \sigma\{\xi_1, \ldots, \xi_i\}$. If $X_i(\omega) = i\xi_i(\omega)$ then

$$\{\tau(\omega) \leq 1\} = \{\omega : \max_{2\leq j} \xi_j(\omega) = -1\} \notin \mathcal{F}_1.$$

Of course it would suffice to take k equal to 2 for the purposes of a counterexample.

*[2] *Suppose $\Omega = (0, 1]$ and $\mathbb{P}$ equals Lebesgue measure restricted to $\mathcal{B}(0, 1]$. Suppose also that μ is another measure on $\mathcal{B}(0, 1]$ for which $\mu B \leq \mathbb{P}B$ for each $B \in \mathcal{B}(0, 1]$. Define $E_{i,k} = \big((i-1)/2^k, i/2^k\big]$ and $\mathcal{E}_k = \{E_{i,k} : 1 \leq i \leq 2^k\}$ and $\mathcal{E} = \cup_{k\in\mathbb{N}_0}\mathcal{E}_k$. Define $\mathcal{F}_n = \sigma(\mathcal{E}_n)$ and*

$$X_k(\omega) = \sum\nolimits_{E\in\mathcal{E}_k} \{\omega \in E\}\frac{\mu E}{\mathbb{P}E}.$$

Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a martingale. Deduce that X_n converges both almost surely and in $\mathcal{L}^1$ to a random variable X for which $\mu B = \mathbb{P}(XB)$ for each $B \in \mathcal{B}(0, 1]$.

SOLUTION: *It was easy to get involved in a lot of unnecessary notation with this problem. The idea was to explain how Radon-Nikodym can be proved using martingales, at least in this simple case.*

It helps to first note that each member of $\mathcal{F}_k$ *is either the empty set or the union of finitely many members of* $\mathcal{E}_k$. *The fact that each* E *in* $\mathcal{E}_k$ *is a union of two sets in* $\mathcal{E}_{k+1}$ *implies that* $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$. *That is, we have a filtration.*

You should check that $\mathcal{B}(0,1] = \mathcal{F}_\infty = \sigma(\mathcal{D})$ *where* $\mathcal{D} = \cup_k \mathcal{F}_k$.

A function f *on* $(0,1]$ *is* $\mathcal{F}_k$*-measurable if and only if it takes a constant value on each* E *in* $\mathcal{E}_k$, *that is,* $f(\omega) = \sum_{E \in \mathcal{E}_k} \{\omega \in E\} f_E$ *for real numbers* $\{f_E : E \in \mathcal{E}_k\}$. *Why? In particular each* X_k *is* $\mathcal{F}_k$*-measurable.*

To prove the martingale property, note that $\mathbb{P}X_k E = \mu E$ *for each* E *in* $\mathcal{E}_k$ *(the* $\mathbb{P}E$ *factors cancel). This equality extends to each* F *in* $\mathcal{F}_k$ *by summing over all the* $\mathcal{E}_k$ *'atoms' in* F. *That is,* $\mathbb{P}X_k F = \mu F$ *for each* $F \in \mathcal{F}_k$.

Remark. *In fact* $X_k = d\mu_k / d\mathbb{P}_k$ *where the subscripts denote restriction of the measures to the sigma-field* $\mathcal{F}_k$

As $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ *we also have*

$$\mathbb{P}X_k F = \mu F = \mathbb{P}X_{k+1} F \qquad \text{for each } F \text{ in } \mathcal{F}_k,$$

which is the desired martingale property. (Why is it enough to check that $\mathbb{P}_{\mathcal{F}_s} X_t = X_s$ *almost surely only for the case* $t = s+1$*?)*

By construction, $0 \le X_k \le 1$. *The convergence theorem for non-negative supermartingales gives* $X_n \to X$ *almost surely for some* $\mathcal{F}_\infty$*-measurable* X. *By Dominated Convergence we also have* $\mathbb{P}|X_n - X| \to 0$.

For each F *in* $\mathcal{D} = \cup_k \mathcal{F}_k$ *there exists a* k *for which* $F \in \mathcal{F}_k$. *For* $n \ge k$ *we have* $\mu F = \mathbb{P}X_k F = \mathbb{P}X_n F$ *so that*

$$|\mu F - \mathbb{P}XF| = |\mathbb{P}X_n F - \mathbb{P}XF| \le \mathbb{P}|X_n - X| \to 0.$$

It follows that $\mu F = \mathbb{P}XF$ *for each* F *in* $\mathcal{D}$. *A generating class argument using the fact that* $\mathcal{D}$ *is a field that generates* $\mathcal{B}(0,1]$ *extends the equality to each* F *in* $\mathcal{B}(0,1]$. *That is,* μ *has density* X *with respect to* $\mathbb{P}$.