Fourier transforms

1 Definitions

S:defn

The Fourier Transform (FT) of a probability measure P on $\mathcal{B}(\mathbb{R})$ is defined as the function

$$\psi_P(t) = P^x e^{itx} \qquad \text{for } t \in \mathbb{R}.$$

It is always well defined because both $\cos(xt)$ and $\sin(xt)$ are bounded continuous functions of x, for each fixed t in $\mathbb{R}$. Moreover, by HW12.5,

$$|\psi_P(t)| \le P^x|e^{itx}| = 1 \qquad \text{for } t \in \mathbb{R}.$$

If X is a (real-valued) random variable then $\psi_X(t)$ is defined to equal the FT of the distribution of X, that is, $\psi_X(t) = \mathbb{P}^\omega e^{itX(\omega)} = \psi_P(t)$ if $X \sim P$.

The FT is closely related to the moment generating function, $M_P(t) = P^x e^{tx} = M_X(T)$ if $X \sim P$. In class I showed, by taking the pointwise limit of the FT for $(X_n - n)/\sqrt{n}$ with $X_n \sim \text{Poisson}(n)$, that

$$\psi_{N(0,1)}(t) = e^{-t^2/2} \qquad \text{for } t \in \mathbb{R}.$$

Compare with $M_X(t) = \mathbb{P}^\omega e^{tX(\omega)} = e^{t^2/2}$ if $X \sim N(0,1)$.

The case where P is the double exponential distribution, with density $e^{-|x|}/2$ with respect to Lebesgue measure is slightly more tricky:

$$\begin{aligned} M_P(t) &= \tfrac12 \int_{\mathbb{R}} e^{-|x|+xt}dx \\ &= \tfrac12 \int_{-\infty}^0 e^{x(1+t)}dx + \tfrac12 \int_0^\infty e^{-x(1-t)}dx \\ &= \begin{cases} (1-t^2)^{-1} & \text{if } |t| < 1 \\ +\infty & \text{otherwise} \end{cases}. \end{aligned}$$

If you have studied complex analysis you might know why the direct replacement of t by it is justified to give $\psi_P(t) = (1+t^2)^{-1}$ for $t \in \mathbb{R}$.

> **Remark.** The function $L(z) := \mathbb{P}^x e^{zx} - (1-z^2)$ is holomorphic in the open set $G = \{z \in \mathbb{C} : -1 < \Re z < 1\}$ and $H(z) = 0$ for $z \in G \cap \mathbb{R}$. By the Corollary to Theorem 10.18 of Rudin (1974), the function L must be zero throughout G.

Even trickier is the case where X has a Cauchy distribution, with density $\pi^{-1}(1+x^2)^{-1}$ with respect to Lebesgue measure.

$$M_X(t) = \begin{cases} 1 & \text{if } t = 0 \\ +\infty & \text{if } t \in \mathbb{R}\backslash\{0\} \end{cases}$$

whereas (Problem [1]) $\psi_X(t) = e^{-|t|}$ for all t in $\mathbb{R}$.

2 Properties of the FT

S:properties

The following facts account for the usefulness of the FT for things like the proof of Central Limit Theorems.

moments

(i) If X is a random variable with $\mathbb{P}|X|^k < \infty$ for some $k \in \mathbb{N}$ then

$$\psi_X(t) = 1 + (it)\mathbb{P}X + \frac{(it)^2}{2!}\mathbb{P}(X^2) + \cdots + \frac{(it)^k}{k!}\mathbb{P}(X^2) + o(|t|^k)$$

near $t = 0$.

unique

(ii) A probability measure P is uniquely determined by its FT.

inversion

(iii) If $\int_{\mathbb{R}} |\psi_P(t)|\,dt < \infty$ then P has a bounded, uniformly continuous density,

$$p(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\psi_P(t)\,dt\,,$$

with respect to Lebesgue measure.

cid

(iv) For probability measures $P, P_1, P_2, \ldots$, if $\psi_{P_n}(t) \to \psi_P(t)$ as $n \to \infty$ for each $t \in \mathbb{R}$ then $P_n \rightsquigarrow P$.

Fact (i) comes from a Taylor expansion of e^{ixt} to a polynomial of degree k plus a remainder that is bounded in modulus by a constant multiple of $|xt|^k \wedge |xt|^{k+1}$, followed by a Dominated Convergence argument.

The other three facts all follow from a strange equality derived using several appeals to Fubini plus invariance arguments for Lebesgue measure.

It turns out to be most convenient to work with the vector space of real-valued function $\mathbb{F} := \text{BL}(\mathbb{R}) \cap \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where λ denotes Lebesgue measure. This space has the following properties:

(a) $\mathbb{F}$ generates $\mathcal{B}(\mathbb{R})$

(b) A probability measure P on $\mathcal{B}(\mathbb{R})$ is uniquely determined by the values $\{Pf : f \in \mathbb{F}\}$.

(c) For probability measures $P, P_1, P_2, \dots$, if $P_n f \to Pf$ for each $f \in \mathbb{F}$ as $n \to \infty$ then $P_n \rightsquigarrow P$.

See Problem [3].

3 A strange identity

S:mu.Q

The identity comes from the fact that there exists a probability measure Q on $\mathcal{B}(\mathcal{R})$ for which $\psi_Q(t) \geq 0$ for each $t \in \mathbb{R}$ and $C \int_{\mathbb{R}} \psi_Q(t)\, dt = 1$ for some constant C. We can then define another probability measure μ on $\mathcal{B}(\mathbb{R})$ by using the density $C\psi_Q(t)$ with respect to Lebesgue measure λ.

For example we could use $Q = N(0,1)$ with $C = (2\pi)^{-1/2}$ and $\mu = N(0,1)$. In my opinion, the dual role of the $N(0,1)$ slightly conceals what is happening. Accordingly, I'll start with a generic pair Q and μ then only impose normality when we come to a bunch of Dominated Convergence arguments. You might find it informative to repeat the argument with Q as the double exponential and μ as the Cauchy.

Let f be a function in $\mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with $\|f\|_1 = \lambda(f)$. Suppose $X \sim P$ independently of $Y \sim \mu$. The joint distribution of (X,Y) is then $P \otimes \mu$. Thus

$$
\begin{aligned}
(\star) = \mathbb{P} f(X+Y) &= P^x \mu^t f(x+t) \\
&= P^x \lambda^t C \psi_Q(t) f(x+t) \qquad \text{definition of } \mu \\
&= C P^x \lambda^t \left(Q^y e^{iyt} f(x+t) \right)
\end{aligned}
$$

By the (analog for three measures of the) Tonelli theorem, the function $(x,y,t) \mapsto e^{iyt} f(x+y)$ is integrable:

$$
P^x \lambda^t Q^y |e^{iyt} f(x+t)| \leq P^x Q^y \int_{\mathbb{R}} |f(x+t)|\, dt = \|f\|_1 < \infty.
$$

Fubini now lets me rearrange the order of integration to get

$$
\begin{aligned}
(\star) &= C Q^y P^x \int_{\mathbb{R}} e^{iyt} f(x+t)\, dt \\
&= C Q^y P^x \int_{\mathbb{R}} e^{iy(w-x)} f(w)\, dw \\
&= C Q^y \int_{\mathbb{R}} e^{iyw} \left(P^x e^{-iyx} \right) f(w)\, dw \\
&= \int_{\mathbb{R}} Q^y \left[C e^{iyw} \psi_P(-y) \right] f(w)\, dw.
\end{aligned}
$$

Actually this is not quite the identity I need. I wrote it that way to make the role of Fubini and invariance of Lebesgue measure clearer. I really need to insert a small positive constant σ to get

mu.QQ.sig <1>
$$\mathbb{P}f(X+\sigma Y) = \int_{\mathbb{R}} H_\sigma(w) f(w)\, dw$$
$$\text{where } H_\sigma(w) = Q^y \left[C e^{iyw/\sigma} \psi_P(-y/\sigma)/\sigma \right]$$

Remark. You could also get the last equality by changing Q to Q_σ, the image of Q under the map $y \mapsto y/\sigma$. For example, if $Q = N(0,1)$ then $Q_\sigma = N(0, 1/\sigma^2)$ and the corresponding μ_σ is the $N(0,\sigma^2)$.

The function $H_\sigma(w)$ is bounded because

$$|H_\sigma(w)| \le Q^y |C e^{iyw/\sigma} \psi_P(-iy/\sigma)/\sigma| \le C/\sigma.$$

In fact (Problem [2]), H_σ must be real-valued with $0 \le H_\sigma \le C/\sigma$ and the distribution of $X + \sigma Y$ has density H_σ with respect to Lebesgue measure.

Now let me specialize to the case $Q = \mu = N(0,1)$ and let me write γ_σ for the distribution of $X + \sigma Y$. Then

identity <2>
$$\mathbb{P}f(X+\sigma Y) = \gamma_\sigma f = \int_{\mathbb{R}} H_\sigma(w) f(w)\, dw \qquad \text{for all } f \in \mathcal{L}^1(\lambda)$$

where

H.sig <3>
$$H_\sigma(w) = \frac{1}{2\pi\sigma} \int_{\mathbb{R}} e^{-y^2/2 + iyw/\sigma} \psi_P(-y/\sigma)\, dy = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\sigma^2 t^2/2 - iyt} \psi_P(t)\, dt.$$

Remember that

H.sig.bnd <4>
$$0 \le H_\sigma(w) \le \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\sigma^2 t^2/2} |\psi_P(t)|\, dt \le \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\sigma^2 t^2/2}\, dt = \frac{1}{\sigma\sqrt{2\pi}}.$$

4 Proofs of Facts (ii), (iii), and (iv)

S:proofs

Specialize equality <2> to f in $\mathbb{F}$.

Fact (ii): *A probability measure P is uniquely determined by its FT.*

Proof The FT ψ_P uniquely determines H_σ for each $\sigma > 0$, which uniquely determines $\mathbb{P}f(X + \sigma Y)$. Let σ tend to zero, invoking Dominated Convergence to see that $\mathbb{P}f(X) = Pf$ is uniquely determined in the limit. Use property (b).

□

Fact (iii): *If* $\int_{\mathbb{R}} |\psi_P(t)|\,dt < \infty$ *then* P *has a bounded, uniformly continuous density,*

$$p(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\psi_P(t)\,dt\,,$$

with respect to Lebesgue measure.

Proof Use the first inequality in <4> to see that the H_σ in <2> stays bounded as as $\sigma \to 0$:

$$0 \le H_\sigma(w) \le \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\sigma^2 t^2/2}|\psi_P(t)|\,dt \le \frac{1}{2\pi}\int_{\mathbb{R}} |\psi_P(t)|\,dt < \infty.$$

Moreover, by Dominated Convergence (with a multiple of $|\psi_P|$ as the dominating function),

$$H_\sigma(w) \to H(w) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iyt}\psi_P(t)\,dt.$$

Again by Dominated Convergence (with a multiple of $|f|$ as the dominating function) and the fact that $f \in \mathbb{F}$,

$$\mathbb{P}f(X) = \lim_{\sigma\to 0}\mathbb{P}f(x+\sigma y) = \int_{\mathbb{R}} H(w)f(w)\,dw.$$

Invoke (b).

□

Fact (iv): *For probability measures* $P, P_1, P_2, \dots,$ *if* $\psi_{P_n}(t) \to \psi_P(t)$ *as* $n \to \infty$ *for each* $t \in \mathbb{R}$ *then* $P_n \rightsquigarrow P$*.*

Proof Suppose $X_n \sim P_n$ independently of $Y \sim N(0,1)$. From the analog of <2> and <5> with X replaced by X_n,

$$\mathbb{P}f(X_n + \sigma Y) = \gamma_\sigma f = \int_{\mathbb{R}} H_{\sigma,n}(w)f(w)\,dw \qquad \text{for all } f \in \mathcal{L}^1(\lambda)$$

where

H.sig <5>
$$H_{\sigma,n}(w) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\sigma^2 t^2/2 - iyt}\psi_{P_n}(t)\,dt.$$

Let n tend to infinity with σ fixed, invoking Dominated Convergence (with a multiple of $\exp(-\sigma^2 w^2/2)$ as the dominating function) using the fact that $\psi_{P_n}(t) \to \psi_P(t)$ for each real t, to deduce that

$$H_{\sigma,n}(w) \to H_\sigma(w) \qquad \text{as } n \to \infty.$$

Then invoke Dominated Convergence (with a multiple of $|f|$ as the dominating function) to deduce that

$$\mathbb{P}f(X_n + \sigma Y) \to \int_{\mathbb{R}} H_\sigma(w) f(w)\, dw = \mathbb{P}f(X + \sigma Y) \qquad \text{as } n \to \infty.$$

Approximate.

$$\begin{aligned} |\mathbb{P}f(X_n) - \mathbb{P}f(X)| \le \mathbb{P}|f(X_n) - f(X_n + \sigma Y)| \\ + |\mathbb{P}f(X_n + \sigma Y) - \mathbb{P}f(X + \sigma Y)| \\ + \mathbb{P}|f(X + \sigma Y) - f(X)| \end{aligned}$$

Let n tend to infinity then use the Bounded Lipschitz property for f to deduce that

$$\limsup_{n\to\infty} |\mathbb{P}f(X_n) - \mathbb{P}f(X)| \le 2\, \|f\|_{\mathrm{BL}}\, \mathbb{P}\left(2 \wedge (\sigma|Y|)\right).$$

Let σ tend to zero then invoke (c) to complete the proof.

□

5 Problems

S:problems

P:Cauchy

[1] Use the inversion fact (iii) from Section 2 together with the FT of the double exponential distribution to show that the Cauchy distribution has FT equal to $e^{-|t|}$.

P:density

[2] Suppose γ is a probability measure on $\mathcal{B}(\mathbb{R})$ and H is a complex-valued, measurable function on $\mathbb{R}$ for which $D := \sup_x |H(x)| < \infty$ and

$$\gamma f = \int_{\mathbb{R}} H(w) f(w)\, dw \qquad \text{for each } f \in \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda).$$

Show that H must be real-valued with $0 \le H(x) \le D$, so that γ has density H with respect to Lebesgue measure. Hint: Let $h_1 = \Re H$ and $h_2 = \Im H$, so that $H(w) = h_1(w) + i h_2(w)$. Show that

$$0 \le \Re\gamma\{h_1 < 0\} = \Re \int_{\mathbb{R}} h_1(w)\{h_1(w) < 0\}\, dw \le 0.$$

Argue similarly to show that $h_2 = 0$.

P:FF

[3] Prove properties (a), (b), and (c) for $\mathbb{F} := \mathrm{BL}(\mathbb{R}) \cap \mathcal{L}^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$. For each $M \in \mathbb{N}$ define

$$J_M(x) := 1 \wedge (M + 1 - |x|)^+ \qquad \text{for } M \in \mathbb{N}.$$

(i) The function J_M belongs to $\mathbb{F}$ and $hJ_M \in \mathbb{F}$ for each $h \in \mathrm{BL}(\mathbb{R})$.

(ii) Show that each bounded interval $[u, v]$ can be written a decreasing limit of a sequence of $\mathbb{F}$ functions. Deduce that $\mathbb{F}$ generates $\mathcal{B}(\mathbb{R})$.

(iii) Show that $P(hJ_M) \to Ph$ as $M \to \infty$ for each $h \in \mathrm{BL}(\mathbb{R})$.

(iv) For (c) show that

$$
\begin{aligned}
P_n(hJ_M) &\to P(hJ_M) \qquad \text{for each } h \in \mathrm{BL}(\mathbb{R}) \text{ and each } M \\
1 - P_n J_M &\to 1 - P J_M.
\end{aligned}
$$

Deduce that

$$
\limsup_{n\to\infty} |P_n f - P f| \le 2 \|h\|_{\mathrm{BL}} P(1 - J_M).
$$

References

Rudin1974complex

Rudin, W. (1974). *Real and Complex Analysis* (Second ed.). New York: McGraw-Hill.