Lambda spaces

<1> **Definition.** *A set $\mathcal{D}$ of subsets of $\mathcal{X}$ is called a* λ***-class of sets*** *if:*

(i) $\mathcal{X} \in \mathcal{D}$

(ii) *$\mathcal{D}$ is stable under proper differences: if $D_1 \supset D_2$ and both sets belong to $\mathcal{D}$ then $D_1 \backslash D_2 \in \mathcal{D}$. Compare with $\mathbb{1}_{D_1 \backslash D_2} = \mathbb{1}_{D_1} - \mathbb{1}_{D_2}$*

(iii) *If $\{D_n : n \in \mathbb{N}\} \subset \mathcal{D}$ and $D_1 \subseteq D_2 \subseteq \dots$ then $D = \cup_{n \in \mathbb{N}} D_n \in \mathcal{D}$. Compare with $\mathbb{1}_{D_n} \uparrow \mathbb{1}_D$ (pointwise).*

<2> **Definition.** *A set $\mathcal{H}$ of bounded, real-valued functions on a set $\mathcal{X}$ is called a* $\boldsymbol{\lambda}$***-space of functions*** *if*

(i) $\mathbb{1} \in \mathcal{H}$

(ii) *$\mathcal{H}$ is a vector space (under pointwise operations)*

(iii) *if $\{h_n : n \in \mathbb{N}\} \subset \mathcal{H}$ and $h_n(x) \uparrow h(x)$ for each $x \in \mathcal{X}$ and the limit h is bounded then $h \in \mathcal{H}$.*

Remark. When checking (iii) there is no loss of generality in assuming $h_1(x) \geq 0$ for all x. Compare with $0 \leq h_n(x) - h_1(x) \uparrow h(x) - h_1(x)$ and the vector space property.

<3> **Definition.** *If $\mathcal{G}$ is a set of real-valued functions on $\mathcal{X}$ then $\sigma(\mathcal{G})$ denotes the smallest sigma-field on $\mathcal{X}$ for which each member of $\mathcal{G}$ is $\sigma(\mathcal{G})\backslash\mathcal{B}(\mathbb{R})$-measurable. It is generated by the collection of all subsets of $\mathcal{X}$ of the form $\{x \in \mathcal{X} : g(x) > t\}$ with $g \in \mathcal{G}$ and $t \in \mathbb{R}$. (Why?)*

If $\mathcal{A}$ is a sigma-field on $\mathcal{X}$, write $\mathcal{M}_{\text{bdd}}(\mathcal{X}, \mathcal{A})$ for the set of all real-valued, bounded, $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable functions on $\mathcal{X}$. Note that $\mathcal{M}_{\text{bdd}}(\mathcal{X}, \mathcal{A})$ is a λ-space. The space $\mathcal{M}_{\text{bdd}}(\mathcal{X}, \sigma(\mathcal{G}))$ plays a role analogous to $\sigma(\mathcal{E})$ for λ-classes of sets.

Key facts

$\mathcal{D}$ a λ-class of sets	$\mathcal{H}$ a lambda space of functions
If $\mathcal{D}$ is stable under pairwise intersections then $\mathcal{D}$ is a sigma-field.	If $\mathcal{H}$ is stable under pairwise products then $\mathcal{H} = \mathcal{M}_{\text{bdd}}(\mathcal{X}, \sigma(\mathcal{H}))$.
If $\mathcal{E}$ is a set of subsets of $\mathcal{X}$ that is stable under pairwise intersections and $\mathcal{D} \supseteq \mathcal{E}$ then $\mathcal{D} \supseteq \sigma(\mathcal{E})$.	If $\mathcal{G}$ is a set of bounded real functions on $\mathcal{X}$ that is stable under pairwise products and $\mathcal{H} \supseteq \mathcal{G}$ then $\mathcal{H} \supseteq \mathcal{M}_{\text{bdd}}(\mathcal{X}, \sigma(\mathcal{G}))$.

Steps towards proving facts for a λ-space $\mathcal{H}$

$\boxed{1}$ (Easy.) Define $\mathcal{D} = \{D \subseteq \mathcal{X} : \mathbb{1}_D \in \mathcal{H}\}$. Show that $\mathcal{D}$ is a λ-class of sets and $\mathcal{D} \subseteq \sigma(\mathcal{H})$. Deduce that $\mathcal{D}$ is a sigma-field if $\mathcal{H}$ is also stable under pairwise products.

$\boxed{2}$ Show that $\mathcal{H}$ is stable under uniform limits. That is, if $h_n \in \mathcal{H}$ and $\|h_n - h\|_\infty := \sup_{x \in \mathcal{X}} |h_n(x) - h(x)| \to 0$ then $h \in \mathcal{H}$. First show that h is uniformly bounded. Then argue that there is a subsequence for which $\left\|h_{n(k)} - h\right\|_\infty \le \delta_k := 2^{-k}$ so that

$$\delta_k + h_{n(k)}(x) \ge h_{n(k-1)}(x) - \delta_{k-1} \qquad \text{for all } x \in \mathcal{X}.$$

Deduce that $g_k(x) := h_{n(k)}(x) + 3\sum_{i \le k} \delta_i$ belongs to $\mathcal{H}$ and it increases pointwise to $h(x) + c$ for some constant c. Then what?

$\boxed{3}$ Suppose $\mathcal{H}$ is stable under pairwise products. Show that if p_n is a polynomial then $p_n \circ h \in \mathcal{H}$ if $h \in \mathcal{H}$. Use the Weierstrass approximation theorem to deduce that if $h \in \mathcal{H}$ takes values in $[a, b]$ and f is a continuous real-valued function on $[a, b]$ then $f \circ h \in \mathcal{H}$. [Remember that $(f \circ h)(x) = f(h(x))$.]

$\boxed{4}$ Deduce from the previous result that if $h, h_1, h_2 \in \mathcal{H}$ then h^+, h^-, $max(h_1, h_2)$, and $\min(h_1, h_2)$ all belong to $\mathcal{H}$.

$\boxed{5}$ Suppose $\mathcal{H}$ is stable under pairwise products. For each $c \in \mathbb{R}$ show that

$$\min\left(1, n(h - c)^+\right) \uparrow \mathbb{1}\{h > c\} \qquad \text{as } n \to \infty.$$

Deduce that $\{h > c\} \in \mathcal{D}$ if $h \in \mathcal{H}$.

$\boxed{6}$ By taking limits of simple functions deduce that $\mathcal{H} \supseteq \mathcal{M}_{\text{bdd}}(\mathcal{X}, \mathcal{D})$ if $\mathcal{H}$ is stable under pairwise products.

$\boxed{7}$ If $\mathcal{G}$ is a set of bounded real functions on $\mathcal{X}$ and $\mathcal{H} \supseteq \mathcal{G}$, show that there is a smallest λ-space $\mathcal{H}_0$ for which $\mathcal{H} \supseteq \mathcal{H}_0 \supseteq \mathcal{G}$. (Mimic the proof for $\mathcal{D}(\mathcal{E})$.)

Write $\mathcal{D}_0$ for $\{D \subseteq \mathcal{X} : \mathbb{1}_D \in \mathcal{H}_0\}$.

Suppose $\mathcal{G}$ is stable under pairwise products.

(i) Mimic the proof of the π–λ theorem for sets to prove that $\mathcal{H}_0$ is stable under pairwise products. Deduce that $\mathcal{D}_0$ is a sigma-field.

(ii) Deduce that $\{h > c\} \in \mathcal{D}_0$ for each h in $\mathcal{H}_0$ and each $c \in \mathbb{R}$.

(iii) Deduce that $\mathcal{D}_0 \supseteq \sigma(\mathcal{G})$. Hint: $\{g > c\} \in \mathcal{D}_0$ for each g in $\mathcal{G}$.

(iv) Deduce that $\mathcal{H}_0 \supseteq \mathcal{M}_{\text{bdd}}(\mathcal{X}, \sigma(\mathcal{G}))$.