Extract from: Henri Lebesgue (1926), "Sur le développement de la notion d'intégrale", Matematisk Tidsskrift B. The copy was made from the English version of the article in the book, entitled "Measure and Integral", edited and translated by Kenneth O. May.

# THE DEVELOPMENT OF THE INTEGRAL CONCEPT

Gentlemen:

Leaving aside all technicalities, we are going to examine the successive modifications and enrichments of the concept of the integral and the appearance of other notions used in recent research on functions of a real variable.

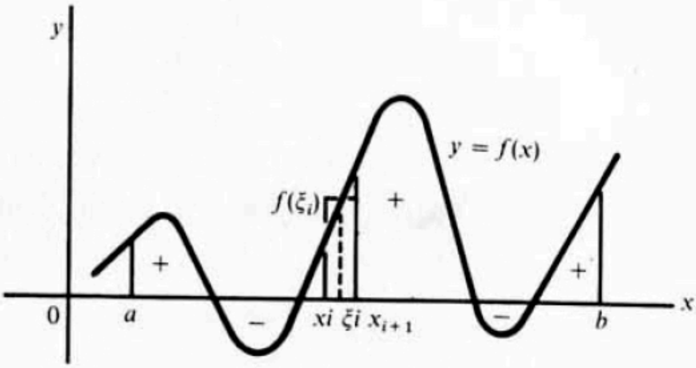
Before Cauchy there was no definition of the integral in the modern meaning of the word "definition." One merely said which areas had to be added or subtracted in order to obtain the integral $\int_a^b f(x)\,dx$.

For Cauchy a definition was necessary, because with him there appeared the concern for rigor which is characteristic of modern mathematics. Cauchy defined continuous functions and their integrals in about the same way as we do today. In order to arrive at the integral of $f(x)$ it suffices to form the sums (Fig. 1)

$$S = \Sigma f(\xi_i)(x_{i+1} - x_i), \tag{1}$$

which surveyors and mathematicians have always used to approximate area, and then deduce the integral $\int_a^b f(x)\,dx$ by passage to the limit.

Although the legitimacy of such a passage to the limit was evident for one who thought in terms of area, Cauchy had to demonstrate that $S$ actually tended to a limit in the conditions he considered. A similar necessity appears every time one replaces an experimental notion by a purely logical definition. One should add that the interest of the defined object




is no longer obvious, it can be developed only from a study of the properties following from the definition. This is the price of logical progress.

What Cauchy did is so substantial that it has a kind of philosophic sweep. It is often said that Descartes reduced geometry to algebra. I would say more willingly that by the use of coordinates he reduced all geometries to that of the straight line, and that the straight line, in giving us the notions of continuity and irrational number, has permitted algebra to attain its present scope.

In order to achieve the reduction of all geometries to that of the straight line, it was necessary to eliminate a certain number of concepts related to geometries of several dimensions such as the length of a curve, the area of a surface, and the volume of a body. The progress realized by Cauchy lies precisely here. After him, in order to complete the arithmetization of mathematics it was sufficient for the arithmeticians to construct the linear continuum from the natural numbers.

And now, should we limit ourselves to doing analysis? No. Certainly, everything that we do can be translated into arithmetical language, but if we renounce direct, geometrical, and intuitive views, if we are reduced to pure logic which does not permit a choice among things that are correct, then we would hardly think of many questions, and certain concepts, for example, most of the ideas that we are going to examine here today, would escape us completely.

For a long time certain discontinuous functions have been integrated. Cauchy's definition still applies to these integrals, but it is natural to examine, as did Riemann, the exact capacity of this definition.

If $\underline{f_i}$ and $\overline{f_i}$ represent the lower and upper bounds of $f(x)$ in $(x_i, x_{i+1})$, then $S$ lies between

$$\underline{S} = \Sigma \underline{f}_i(x_{i+1} - x_i) \qquad \text{and} \qquad \overline{S} = \Sigma \overline{f}_i(x_{i+1} - x_i).$$

Riemann showed that for the definition of Cauchy to apply it is sufficient that

$$\overline{S} - \underline{S} = \Sigma(\overline{f}_i - \underline{f}_i)(x_{i+1} - x_i)$$

tends toward zero for a particular sequence of partitions of the interval from $a$ to $b$ into smaller and smaller subdivisions $(x_i, x_{i+1})$. Darboux added that under the usual operation of passage to the limit $\underline{S}$ and $\overline{S}$ always give two definite numbers

$$\underline{\int_a^b} f(x)\,dx \qquad \text{and} \qquad \overline{\int_a^b} f(x)\,dx.$$

These numbers are generally different and are equal only when the Cauchy-Riemann integral exists.

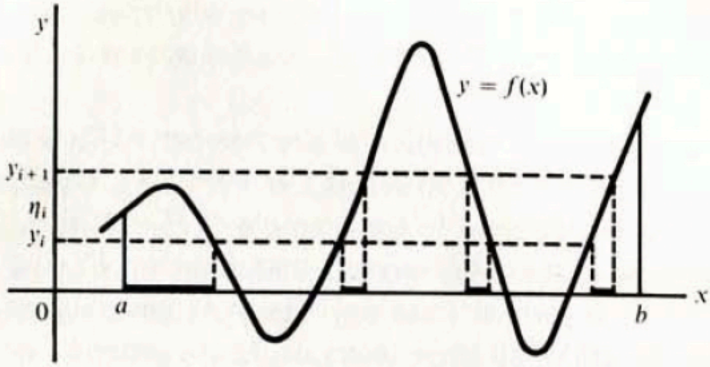
From a logical point of view, these are very natural definitions, aren't they? However, one can say that from a practical point of view they have been useless. In particular, Riemann's definition has the drawback of applying only rarely and in a sense by chance.

It is evident that breaking up the interval $(a, b)$ into smaller and smaller subintervals $(x_i, x_{i+1})$ makes the differences $\overline{f}_i - \underline{f}_i$ smaller and smaller if $f(x)$ is continuous, and that the continued refinement of the subdivision will make $\overline{S} - \underline{S}$ tend toward zero if there are only a few points of discontinuity. But we have no reason to hope that the same thing will happen for a function that is discontinuous everywhere. To take smaller intervals $(x_i, x_{i+1})$, that is to say values of $f(x)$ corresponding to values of $x$ closer together, does not in any way guarantee that one takes values of $f(x)$ whose differences become smaller.

Let us be guided by the goal to be attained—to collect approximately equal values of $f(x)$. It is clear then that we must break up not $(a, b)$, but the interval $(\underline{f}, \overline{f})$ bounded by the lower and upper bounds of $f(x)$ in $(a, b)$. Let us do this with the aid of number $y_i$ differing among themselves by less than $\epsilon$. We are led to consider the values of $f(x)$ defined by

$$y_i \leq f(x) \leq y_{i+1}.$$

The corresponding values of $x$ form a set $E_i$. In Figure 2 this set $E_i$ consists of four intervals. With some continuous functions it might consist of an infinity of intervals. For an arbitrary function it might be very complicated. But this matters little. It is this set $E_i$ which plays the role analogous to the interval $(x_i, x_{i+1})$ in the usual definition of the integral




of continuous functions, since it tells us the values of $x$ which give to $f(x)$ approximately equal values.

If $\eta_i$ is any number whatever taken between $y_i$ and $y_{i+1}$, $y_i \leq \eta_i \leq y_{i+1}$, the values of $f(x)$ for points of $E_i$ differ from $\eta_i$ by less than $\epsilon$. The number $\eta_i$ is going to play the role which $f(\xi_i)$ played in formula (1). As to the role of the length or measure $x_{i+1} - x_i$ of the interval $(x_i, x_{i+1})$, it will be played by a measure $m(E_i)$ which we shall assign to the set $E_i$ in a moment. In this way we form the sum

$$S = \Sigma \eta_i m(E_i). \tag{2}$$

Let us look closely at what we have just done and, in order to understand it better, repeat it in other terms.

The geometers of the seventeenth century considered the integral of $f(x)$—the word "integral" had not been invented, but that does not matter—as the sum of an infinity of indivisibles, each of which was the ordinate, positive or negative, of $f(x)$. Very well! We have simply grouped together the indivisibles of comparable size. We have, as one says in algebra, collected similar terms. One could say that, according to Riemann's procedure, one tried to add the indivisibles by taking them in the order in which they were furnished by the variation in $x$, like an unsystematic merchant who counts coins and bills at random in the order in which they came to hand, while we operate like a methodical merchant who says:

I have $m(E_1)$ pennies which are worth $1 \cdot m(E_1)$,
I have $m(E_2)$ nickels worth $5 \cdot m(E_2)$,
I have $m(E_3)$ dimes worth $10 \cdot m(E_3)$, etc.

Altogether then I have

$$S = 1 \cdot m(E_1) + 2 \cdot m(E_2) + 5 \cdot m(E_3) + \cdots.$$

The two procedures will certainly lead the merchant to the same result because no matter how much money he has there is only a finite number of coins or bills to count. But for us who must add an infinite number of indivisibles the difference between the two methods is of capital importance.

We now consider the definition of the number $m(E_i)$ attached to $E_i$. The analogy of this measure to length, or even to a number of coins, leads us naturally to say that, in the example of Fig. 2, $m(E_i)$ will be the sum of the lengths of the four intervals that make up $E_i$, and that, in an example where $E_i$ is formed from an infinity of intervals, $m(E_i)$ will be the sum of the length of all these intervals. In the general case it leads us to proceed as follows. Enclose $E_i$ in a finite or denumerably infinite number of intervals, and let $l_1, l_2, \ldots$ be the length of these intervals. We obviously wish to have

$$m(E_i) \leq l_1 + l_2 + \cdots.$$

If we look for the greatest lower bound of the second member for all possible systems of intervals that cover $E_i$, this bound will be an upper bound of $m(E_i)$. For this reason we represent it by $\overline{m(E_i)}$, and we have

$$m(E_i) \leq \overline{m(E_i)}. \tag{3}$$

If $C$ is the set of points of the interval $(a, b)$ that do not belong to $E_i$, we have similarly

$$m(C) \leq \overline{m(C)}.$$

Now we certainly wish to have

$$m(E_i) + m(C) = m[(a, b)] = b - a;$$

and hence we must have

$$m(E_i) \geq b - a - \overline{m(C)}. \tag{4}$$

The inequalities (3) and (4) give us upper and lower bounds for $m(E_i)$. One can easily see that these two inequalities are never contradictory. When the lower and upper bounds for $E_i$ are equal, $m(E_i)$ is defined, and we say then that $E_i$ is measurable.[1]

[1] The definition of measure of sets used here is that of C. Jordan, *Cours d'analyse de l'École Polytechnique*, Vol. I, but with this modification, essential for our purpose, that we enclose the set $E_i$ to be measured in intervals whose number may be infinite, while Jordan employed only a finite number. This use of a denumerable infinity in place of a finite number of intervals was suggested by the work of Borel, who himself had utilized this idea in order to get a definition of measure (*Leçons sur la théorie des fonctions*).