# MISCELLANEA

## Studies in the history of probability and statistics
## VI. A note on the early solutions of the problem of the duration of play

By A. R. THATCHER

It is now just 300 years since the publication by Huygens of the first result on the famous problem which became known as the Duration of Play. The aim of this note is to summarize the early development of this problem and to show how easily some of the solutions found at the beginning of the eighteenth century can be linked with modern work on sequential tests, random walks and certain storage problems.

We use throughout the following notation. Call the two players $A$ and $B$, and let their chances of winning a game be $p$ and $q = 1-p$, respectively. $A$ starts with $a$ counters and $B$ starts with $b$ counters, and after each game the loser hands one counter to the winner. It is desired to find first the probability $P_a$ that $A$ will eventually lose all his counters without having previously won all $B$'s, and more generally the probability $P_{a,n}$ that this will happen within $n$ games. $P_b$ and $P_{b,n}$ are defined similarly. $P_{a,n}+P_{b,n}$ is the probability that the play will terminate (with the 'ruin' of one of the players) within $n$ games. It can be shown that the play must end sooner or later, so that $P_a+P_b = 1$.

In 1657 Huygens gave without proof, in the fifth and last problem of his treatise *De ratiociniis in ludo aleae*, the numerical value for $P_a$ in a case where $a = b = 12$ and where $p$ and $q$ had particular values. The general result for $P_a$ was found by James Bernoulli, who died in 1705, but it remained in manuscript until it was published 8 years later in his *Ars Conjectandi*; Bernoulli says that the proof is laborious and leaves it to the reader. Before the *Ars Conjectandi* appeared, however, de Moivre had found a simple derivation independently and published it in his treatise *De Mensura Sortis* (1711).

De Moivre's original proof, which was later reproduced in his *Doctrine of Chances* (see 1711, pp. 227–8; 1718, pp. 23–4; 1738, pp. 45–6; 1756, pp. 52–3), is very ingenious and so much shorter than the demonstrations usually given in modern textbooks that it is worth quoting. Its essence is as follows. Imagine that each player starts with his counters before him in a pile, and that nominal values are assigned to the counters in the following manner. $A$'s bottom counter is given the nominal value $q/p$; the next is given the nominal value $(q/p)^2$, and so on until his top counter which has the nominal value $(q/p)^a$. $B$'s top counter is valued $(q/p)^{a+1}$, and so on downwards until his bottom counter which is valued $(q/p)^{a+b}$. After each game the loser's top counter is transferred to the top of the winner's pile, and it is always the top counter which is staked for the next game. Then *in terms of the nominal values* $B$'s stake is always $q/p$ times $A$'s, so that at every game each player's nominal expectation is nil. This remains true throughout the play; therefore $A$'s chance of winning all $B$'s counters, multiplied by his nominal gain if he does so, must equal $B$'s chance multiplied by $B$'s nominal gain. Thus

$$P_b\left\{\left(\frac{q}{p}\right)^{a+1}+\left(\frac{q}{p}\right)^{a+2}+\ldots+\left(\frac{q}{p}\right)^{a+b}\right\} = P_a\left\{\left(\frac{q}{p}\right)+\left(\frac{q}{p}\right)^{2}+\ldots+\left(\frac{q}{p}\right)^{a}\right\}.$$

The use of $P_a+P_b = 1$ now gives immediately

$$P_b = \frac{(q/p)^a-1}{(q/p)^{a+b}-1}, \tag{1}$$

and this is the probability of the 'gambler's ruin'.

In terms of the counters, $A$'s total expected gain is $bP_b-aP_a$, while his expectation per game is $p-q$. These obvious facts are indeed only special cases of a more general result given by de Moivre (1718, pp. 135–6; 1738, pp. 48–9; 1756, pp. 55–6). De Moivre does not actually divide one expression by the other, but, since the total expectation equals the expectation per game times the expected number of games, this division is all that is required in order to get the expected number of games

$$E(N) = \frac{bP_b-aP_a}{p-q}. \tag{2}$$

De Moivre was also the first to discover and publish a general method for calculating $P_{a,n}+P_{b,n}$, thus finding the chance that the play would terminate within $n$ games. For the case where $a$ is infinite (so that $P_{a,n} = 0$) and $n-b$ is odd, he found

$$P_{b,n} = \text{first } \tfrac{1}{2}(n-b+1) \text{ terms of } (p+q)^n + \text{first } \tfrac{1}{2}(n-b+1) \text{ terms of } (p/q)^b(q+p)^n. \tag{3}$$

This solution, with a similar one for the case where $n-b$ is even, was given without proof in his *De Mensura Sortis* and later in *The Doctrine of Chances* (1711, p. 262; 1718, pp. 119–20; 1738, p. 179; 1756, pp. 208–9). Fieller (1931) has drawn attention to this result and also provided a simple and elegant proof.

De Moivre's first solution of the general problem of calculating $P_{a,n}+P_{b,n}$ when both $a$ and $b$ are finite (1711, p. 261; 1718, pp. 113–14; 1738, stated incorrectly on pp. 173–4; 1756, p. 203) called for the performance of $n-1$ multiplications and the rejection of certain terms during the process. For moderate $n$ the calculation is not so tedious as appears at first sight, and it has the advantage of giving the answer reduced to the smallest number of terms; as de Moivre later pointed out, the rejected terms can also be used to obtain $P_{a,n}$ and $P_{b,n}$ separately.

However, a few months before de Moivre's method actually appeared (for the *Philosophical Transactions* for 1711 were delayed in the press), a different solution giving $P_{a,n}$ and $P_{b,n}$ separately had been found and was soon published by de Montmort (1713). This result is of particular interest because it provides one of the easiest solutions of the problem, since the series which can be derived from it by using modern tables is rapidly convergent over the range of values of $n$ where the play is likely to terminate.

In 1710 de Montmort found a method for calculating $P_{a,n}$ and $P_{b,n}$ for the case $p = q$. He sent some numerical results to John Bernoulli, who passed the letter to his nephew Nicholas. In a reply dated 26 February 1711, published by de Montmort (1713, p. 308 et seq.), Nicholas Bernoulli gave without proof the general solution for the case $p \neq q$; in modern notation it can be written as follows:

$$P_{b\,n} = \sum_t \left\{p^{ts+b}q^{ts}\sum_i \binom{n}{i}(p^{n-b-2ts-i}q^i+q^{n-b-2ts-i}p^i)\right\}$$
$$-\sum_t \left\{p^{ts+s}q^{ts+a}\sum_i \binom{n}{i}(p^{n-b-2ts-2a-i}q^i+q^{n-b-2ts-2a-i}p^i)\right\}. \quad (4)$$

In this formula $s = a+b$; the summation over $i \geqslant 0$ continues until the terms in the series in each curly bracket, re-arranged in descending powers of $p$, meet in the middle (the middle term counting only once if $n-b$ is even); and the summation over $t$ covers all values $\geqslant 0$ which leave non-negative exponents within the summation over $i$ on the line concerned. Bernoulli stated the result for $n-b$ even, but in fact (4) is also valid if $n-b$ is odd.

Not content with this, Nicholas Bernoulli confirmed that the limit of (4) as $n \to \infty$ gives the correct value for $P_b$. He does not give his method but it is not difficult to guess; if for example $p > q$ it is only necessary to re-write the two lines of (4) as

$$\sum_t p^{-ts}q^{ts}[p^n+np^{n-1}q+\ldots+p^{2ts+b}q^{n-2st-b}]$$
$$-\sum_t p^{-ts-a}q^{ts+a}[p^n+np^{n-1}q+\ldots+p^{2ts+2a+b}q^{n-2ts-2a-b}]. \quad (5)$$

As $n \to \infty$ the sums in each square bracket tend to 1; this follows from (James) Bernoulli's Theorem, which at the time had not been published but which was known to Nicholas. The expression thus reduces to two geometric series, and is immediately seen to agree with (1) above. In passing, it may be noted that as $a \to \infty$ the expression (4) reduces to de Moivre's expression (3).

When de Montmort saw this extraordinary solution he admitted that he could not follow it (this was partly because Bernoulli had inadvertently used one symbol in two senses), and remarked: 'votre formule m'étonne pour sa generalité' (1713, p. 316). Later, in comparing it with his own, he said: 'je n'ai eu en vûe que la supposition des hazards égaux pour l'un et pour l'autre Joueur, au lieu que vous les supposés dans un rapport quelconque' (1713, p. 345). De Montmort's solution, which he then describes briefly, consisted of a method of picking out the binomial coefficients in (4) from Pascal's triangle; this was of course sufficient when $p = q$, and was in itself a remarkable result to have found. Nevertheless, it seems clear that the solution (4) of the general case $p \neq q$, though often described as de Montmort's, was in fact found first by Nicholas Bernoulli.

De Montmort reproduced (4) in the body of his book, gave an example and added a most interesting though far from rigorous demonstration (1713, pp. 268–72). De Moivre at first called the result 'very handsom' (1718, p. 122), but later criticized de Montmort's statement of it (which indeed is not entirely correct) and seems to hint that he had found the same method of solution before the year 1711 (see 1738, pp. 181–2; 1756, pp. 210–11). This is certainly possible, though it may be doubted whether de Moivre had carried the investigation of (4) as far as Bernoulli; perhaps he used it in particular cases, but did not pursue the matter because his own result gave $P_{a,n}+P_{b,n}$ in a smaller number of terms.

De Moivre later solved the Duration of Play problem in two further ways, and in the course of his work made an extensive investigation of recurring series (which he was the first to explore). His results included the partial fraction expansion of a generating function (1738, pp. 197–99; 1756, pp. 224–7); he found the probability of runs of successes (1738, pp. 243–8; 1756, pp. 254–9), and of course made the original derivation of the normal distribution (1738, pp. 235–43; 1756, pp. 243–50). On the Duration of Play problem itself he expressed $P_{b,n}$ as a recurring series with fewer terms than (4); and finally he discovered the first results on the trigonometrical solution (see Feller, 1950, p. 292, equation 5·7), including the asymptotic form for $P_{b,n}$ when $a = b$ and $p = q$. For fuller details of his work, and of its subsequent development by Laplace and many others, the reader is referred to Todhunter (1865) and Fieller (1931).

It remains to show the link between these early solutions and modern work. This stems from the well-known fact that the Duration of Play situation can be regarded as a linear random walk with two absorbing barriers, such that the movement of the particle at each jump has a distribution with mean $\mu = p-q$ and variance $\sigma^2 = 4pq$. To complete the comparison a simple approximation is required, namely

$$(p/q)^{\lambda} \simeq \exp(2\lambda\mu/\sigma^2), \tag{6}$$

which can be shown to apply with sufficient accuracy in the cases for which it will be required.

If then in equations (1) and (2) we make the substitutions (6) and $p-q = \mu$ we shall obtain approximations for the probability of absorption at a given barrier, and for the expected number of steps before absorption at either barrier, in the corresponding random walk; and under the conditions of the central limit theorem these will be valid for all walks with given finite $\mu$ and $\sigma$, provided that the number of steps is sufficiently large. It can be seen by inspection that the transformed version of equations (1) and (2) are in fact the same as Wald's approximations for the operating characteristic and average sample number of a sequential test, in the form quoted by Page (1954, equations 5, 7).

We can similarly transform (3), making the normal approximation to the binomial expressions; it will be found that the result agrees with that given by Bartlett (1946, equation 8), obtained as the solution of a differential equation for the diffusion process. It is of interest to note that the same result can also be used to find a quick approximate solution of a storage problem considered in a recent paper by Anis (1956). This concerns a reservoir, of unlimited capacity, which has initial water level $x$; this level varies each year by an amount distributed with zero mean and unit variance. When $n$ and $x$ are sufficiently large we can ignore the end-effects and assume that the probability that the reservoir will run dry within $n$ years is approximately the same as the probability that $B$ will lose $b = x$ counters within $n$ trials (where $a$ is infinite and $p = q = \frac{1}{2}$). By de Moivre's result (3) this probability will be twice the sum of the first $\frac{1}{2}(n-x+1)$ terms of $(\frac{1}{2}+\frac{1}{2})^n$. Hence, for large $n$ and $x$ the probability that the reservoir will not run dry within $n$ years can be expressed approximately as $2\int_0^{x/\sqrt{n}} e^{-\frac{1}{2}t^2}/\sqrt{(2\pi)}\,dt$, and it is easy to verify that this distribution has the same moment ratios as the limiting values found by Anis.

Finally, we come to Nicholas Bernoulli's general solution of the Duration of Play. If for any value of $t$ either line of (4) is arranged in descending powers of $p$, it will be found to be the sum of multiples of two binomial expressions in the same way as (3)—see also Fieller (1931, equation 10.1), who proceeds to obtain the exact solution of the problem in a convenient form as a series of multiples of incomplete beta-functions, and also provides a rigorous proof.

The application of (6) and the normal approximation to the binomial puts the solution in the simple approximate form $\Sigma A_i \int_{a_i}^{\infty} \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}x^2}\,dx$; this series agrees with the (exact) result given by Bartlett (1946, equation 17) for the diffusion process. In view of the usefulness of this series it is worth repeating here for completeness

$$P_{b,n} \simeq F(b) - w(-a)\,F(b+2a) + w(-a-b)\,F(3b+2a) - w(-2a-b)\,F(3b+4a) + \dots, \tag{7}$$

where

$$F(\lambda) \equiv Q\left(\frac{\lambda}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right) + w(\lambda)\,Q\left(\frac{\lambda}{\sigma\sqrt{n}} + \frac{\mu\sqrt{n}}{\sigma}\right),$$

$$Q(\lambda) \equiv \int_{\lambda}^{\infty} \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}x^2}\,dx,$$

$$w(\lambda) \equiv \exp(2\lambda\mu/\sigma^2).$$

The corresponding series for $P_{a,n}$ is found by interchanging $a$ with $b$ and changing the sign of $\mu$ in the definitions of $F$ and $w$.

It will be found that (7) converges rapidly over the range of $n$ where the process is likely to terminate, and so (as suggested by Bartlett) provides a rapid approximation for the probability that a particle starting at the origin, with a jump distribution having mean $\mu$ and variance $\sigma^2$, will reach $x = b$ (without having previously been absorbed at $x = -a$) within $n$ jumps. It can similarly be used to find the chance that a linear sequential test will end within $n$ trials, or that a finite reservoir with random net input will either dry up or overflow within a given time.

REFERENCES

ANIS, A. A. (1956). *Biometrika*, **43**, 79.
BARTLETT, M. S. (1946). *Proc. Camb. Phil. Soc.* **42**, 239.
DE MOIVRE, A. (1711). De Mensura Sortis. *Phil. Trans.* **27**, 213.
DE MOIVRE, A. (1718). *The Doctrine of Chances*, 1st ed. London.
DE MOIVRE, A. (1738). *The Doctrine of Chances*, 2nd ed. London.
DE MOIVRE, A. (1756). *The Doctrine of Chances*, 3rd ed. London.
DE MONTMORT, P. R. (1713). *Essai d'Analyse sur les Jeux de Hazard*, 2nd ed. Paris.
FELLER, W. (1950). *An Introduction to Probability Theory and its Applications.* New York: Wiley.
FIELLER, E. C. (1931). *Biometrika*, **22**, 377.
PAGE, E. S. (1954). *J. R. Statist. Soc.* B, **16**, 136.
TODHUNTER, I. (1865). *History of the Theory of Probability.* Cambridge and London: Macmillan.

## Optimal sampling for quota fulfilment

BY N. L. JOHNSON
*University College London*

1. The problem to be discussed in this paper arises in the following way. It is desired to obtain a sample from a stratified population in such a way that there are exactly $m_i$ individuals from stratum $\omega_i (i = 1, \ldots, k)$. It is more convenient to take a random sample from the whole population, and to ascertain subsequently the strata to which the chosen individuals belong, than to search for individuals belonging to specified strata. Therefore, a first sample of $N$ individuals is chosen without regard to stratification and any shortfall is made up by a further set of samples, each restricted to one of the deficient strata, and of such a size as to provide the required number of individuals from each of the strata. Thus, if the first sample of $N$ contains $n_i (< m_i)$ individuals from stratum $\omega_i$ then the subsequent sample from this stratum will contain $m_i - n_i$ individuals but if $n_i \geqslant m_i$, no subsequent sample from this stratum will be required.

If $c$ is the cost per individual in the first (unrestricted) sample, and $c_i$ the cost per individual for a sample restricted to stratum $\omega_i$, then the expected cost of obtaining the complete sample is

$$C_1 = cN + \sum_{i=1}^{k} c_i \mathscr{E}(m_i - n_i \mid n_i < m_i) \Pr\{n_i < m_i\}, \tag{1}$$

where $n_i$ is the number of observations included in $\omega_i$ in the first unrestricted sample. If the unused individuals in stratum $\omega_i$ with numbers in excess of requirements are worth $c_i'$ each the expected cost is

$$C_2 = cN + \sum_{i=1}^{k} [c_i \mathscr{E}(m_i - n_i \mid n_i < m_i) \Pr\{n_i < m_i\} + c_i' \mathscr{E}(m_i - n_i \mid n_i \geqslant m_i) \Pr\{n_i \geqslant m_i\}]. \tag{2}$$

$C_1$ can, of course, be regarded as a special case of $C_2$.

2. If it is supposed that the joint distribution of $n_1, n_2, \ldots, n_k$ is multinomial with parameters $p_1, p_2, \ldots, p_k$ (as would be appropriate if sampling from a large population with proportions $p_1, p_2, \ldots, p_k$ in strata $\omega_1, \omega_2, \ldots, \omega_k$, respectively, were being considered) then

$$\mathscr{E}(m_i - n_i) = m_i - Np_i$$

and (2) can be written

$$C_2 = cN + \sum_{i=1}^{k} (c_i - c_i') \mathscr{E}(m_i - n_i \mid n_i < m_i) \Pr\{n_i < m_i\} + \sum_{i=1}^{k} c_i'(m_i - Np_i). \tag{2a}$$

Using Gruder's formula $\displaystyle\sum_{r=0}^{m-1} (r - Np) \binom{N}{r} p^r q^{N-r} = -m \binom{N}{m} p^m q^{N-m+1},$

$C_2$ can be expressed in the form

$$C_2 = cN + \sum_{i=1}^{k} (c_i + c_i') \left[ (m_i - Np_i) \sum_{j=0}^{m_i-1} \binom{N}{j} p_i^j q_i^{N-j} + m_i \binom{N}{m_i} p_i^{m_i} q_i^{N-m_i+1} \right] + \sum_{i=1}^{k} c_i'(m_i - Np_i). \tag{3}$$