Convergent subsequences

1 Complete metric spaces

A metric space (M, d) is said to be complete if every Cauchy sequence is convergent to a point in the space. Remember that $\{m_i : i \in \mathbb{N}\}$ is a Cauchy sequence in M if for each $\delta > 0$ there exists an $n_\delta \in \mathbb{N}$ for which

<1> $$d(m_i, m_j) < \delta \qquad \text{for all } i, j \geq n_\delta.$$

The prime example of a complete metric space is the real line, under its usual metric.

Notice that <1> refers to all pairs i, j with $\min(i, j) \geq n_\delta$, not just pairs with $j = i + 1$.

The $\mathcal{L}^p$ spaces defined in UGMTP Section 2.7 are also complete under the metric defined by the norm $\|\cdot\|_p$. For example, if $\{f_i : i \in \mathbb{N}\} \subset \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ and, for each $\delta > 0$

<2> $$\|f_i - f_j\|_1 < \delta \qquad \text{for } \min(i, j) \geq n_\delta$$

then there exists an f in $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ for which $\|f_i - f\|_1 \to 0$ as $i \to \infty$.

Remark. Technically speaking, $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ is only a semi-metric space and $\|\cdot\|_1$ is only a semi-norm, because $\|f\|_1 = 0$ implies not that f is everywhere zero but rather that f is zero μ-almost everywhere. The technical fix is to work with the set $L^1(\mathcal{X}, \mathcal{A}, \mu)$ of μ-equivalence classes of functions from $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$.

The proofs given in UGMPTP Problems 2.18, 2.19, and 2.22 can be simplified by means of an extension of Problem 2.14 to general measures, which appears in the next Section.

2 Cauchy sequences in measure

Suppose $\{f_n : n \in \mathbb{N}\}$ is a sequence of real-valued $\mathcal{B}(\mathcal{R})$-measurable functions on a measure space $(\mathcal{X}, \mathcal{A}, \mu)$ for which: for each $\epsilon > 0$ and $\delta > 0$ there exists an $n_{\epsilon,\delta}$ for which

<3> $$\mu\{x : |f_n(x) - f_m(x)| > \delta\} < \epsilon \qquad \text{whenever } m, n \geq n_{\epsilon,\delta}.$$

Then there exists a subsequence $\{f_{n(k)} : k \in \mathbb{N}\}$ and a real-valued $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable function f for which

$$f_{n(k)}(x) \to f(x) \qquad \text{as } k \to \infty, \text{ for } \mu\text{-almost all } x \text{ in } \mathcal{X}.$$

Remark. A sequence satisfying <3> is sometimes said to be a Cauchy sequence in measure.

Proof Write $n(k)$ for the $n_{\epsilon,\delta}$ corresponding to $\epsilon = \delta = 2^{-k}$, for $k \in \mathbb{N}$. You will see soon why I need both the δ and ϵ sequences to be summable.

We may assume that $n(1) < n(2) < \dots$, to avoid trivial situations such as $f_n(x) = 0$ for all x and n, in which case we could have $n(\delta, \epsilon) = 1$ for all $\delta > 0$ and $\epsilon > 0$.

Remark. More formally we could define $n(1) = n(1/2, 1/2)$ then $n(2) = \max(1 + n(1), n(1/4, 1/4)$, and so on. Very fussy.

By construction,

$$\mu\{|f_i(x) - f_j(x)| > 2^{-k}\} \leq 2^{-k} \qquad \text{when } \min(i, j) \geq n(k).$$

In particular,

$$\mu\{|f_{n(k)}(x) - f_{n(k+1)}(x)| > 2^{-k}\} \leq 2^{-k}.$$

Thus

$$\sum\nolimits_{k\in\mathbb{N}} \widetilde{\mu}\mathbb{1}\{x : |f_{n(k)}(x) - f_{n(k+1)}(x)| > 2^{-k}\} \leq \sum\nolimits_{k\in\mathbb{N}} 2^{-k} < \infty.$$

That is the reason for choosing a summable ϵ sequence.

By Monotone Convergence we can move the $\widetilde{\mu}$ outside the summation then deduce existence of a μ-negligible set N for which

$$\sum\nolimits_{k\in\mathbb{N}} \mathbb{1}\{x : |f_{n(k)}(x) - f_{n(k+1)}(x)| > 2^{-k}\} < \infty \qquad \text{for all } x \text{ in } N^c.$$

For each x in N^c only finitely many of the indicator functions can take the value 1. Of course the number of such terms can be different for every x. That is, for each x in N^c there exists a $k_0(x) \in \mathbb{N}$ for which

$$|f_{n(k)}(x) - f_{n(k+1)}(x)| \leq 2^{-k} \qquad \text{for all } k \geq k_0(x).$$

Remark. Note that we have no reason to hope $\sup_{x\in N^c} k_0(x)$ is also finite. I often find such an assertion in homework solutions.

To simplify notation let me write $g_k(x)$ for $f_{n(k)}(x)$. Then for each x in N^c and all i and j with $k_0(x) \le i < j$ we have

$$\begin{aligned} |g_i(x) - g_j(x)| &= |\sum\nolimits_{k=i}^{j-1} g_k(x) - g_{k+1}(x)| \\ &\le \sum\nolimits_{k=i}^{j-1} |g_k(x) - g_{k+1}(x)| \\ &\le \sum\nolimits_{k=i}^{j-1} 2^{-k} < 2^{1-i}. \end{aligned}$$

For each $\delta > 0$ we have $|g_i(x) - g_j(x)| < \delta$ if $\min(i, j)$ is large enough. That is, for each $x \in N^c$ the real numbers $\{g_k(x) : k \in \mathbb{N}\}$ form a Cauchy sequence.

By completeness of the real line, for each x in N^c we know that $g_k(x)$ converges to some real number $f(x)$. For x in N we know nothing, so it is perhaps wise to define $f(x) = 0$ if $x \in N$. The function f is then defined everywhere and, by virtue of the equality

$$f(x) = \limsup_k \left(g_k(x)\mathbb{1}\{x \in N^c\}\right) \qquad \text{for each } x,$$

it is $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable. The limsup is actually a limit on N^c. We have

$$f_{n(k)}(x) = g_k(x) \to f(x) \qquad \text{a.e. } [\mu].$$

□

3 Completeness of $\mathcal{L}^1$

As in <2>, suppose $\{f_i : i \in \mathbb{N}\} \subset \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ and, for each $\delta > 0$

$$\|f_i - f_j\|_1 < \delta \qquad \text{for } \min(i, j) \ge n_\delta.$$

Given $\gamma > 0$ and $\epsilon > 0$ we have

$$\mu\{|f_i - f_j| > \delta\} \le \mu|f_i - f_j|/\delta < \epsilon \qquad \text{if } \min(i, j) \ge n_\gamma \text{ with } \gamma = \epsilon\delta.$$

That is, the sequence is also Cauchy in measure, in the sense defined by <3>.

From Section 2, there exists a real-valued, $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable function f for which there is a subsequence with

$$f_{n(k)} \to f \qquad \text{a.e. } [\mu] \text{ as } k \to \infty.$$

We also have

$$\delta > \widetilde{\mu}|f_n - f_{n(k)}| \qquad \text{provided } n, n(k) \ge n_\delta.$$

With n held fixed let k tend to infinity. By Fatou, for $n \ge n_\delta$,

$$\delta \ge \liminf_{k\to\infty} \widetilde{\mu}|f_n - f_{n(k)}| \ge \widetilde{\mu} \liminf_{k\to\infty} |f_n - f_{n(k)}| = \widetilde{\mu}|f_n - f|$$

because $|f_n - f_{n(k)}| \to 0$ a.e. $[\mu]$. The function f belongs to $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ because $\widetilde{\mu}|f| \le \widetilde{\mu}|f_n - f| + \widetilde{\mu}|f_n|$.