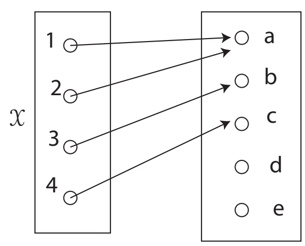
Statistics 330b/600b, Math 330b spring 2017
Homework # 1
Due: Thursday 25 January

Please attempt at least the starred problems. Please explain your reasoning. Please look at the handout latex.pdf for an explanation of why solutions consisting of a long sequence of ∴'s and ∵'s can be hard to follow.

*[1] Suppose T maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For each $B \subseteq \mathcal{Y}$ and $A \subseteq \mathcal{X}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$ and $T(A) := \{T(x) : x \in A\}$. In class I asserted that

(i) $T^{-1}\left(\cup_i B_i\right) = \cup_i T^{-1}(B_i)$
(ii) $T^{-1}\left(\cap_i B_i\right) = \cap_i T^{-1}(B_i)$
(iii) $T^{-1}\left(B^c\right) = \left(T^{-1}\left(B\right)\right)^c$



In my experience, many students also believe that

(i) $T\left(\cup_i A_i\right) = \cup_i T(A_i)$
(ii) $T\left(\cap_i A_i\right) = \cap_i T(A_i)$
(iii) $T\left(A^c\right) = \left(T\left(A\right)\right)^c$
(iv) $T^{-1}\left(T(A)\right) = A$
(v) $T\left(T^{-1}(B)\right) = B$.

In general, some of these assertions are false. Provide counterexamples for each of the false assertions. Maybe you could also give extra conditions under which the assertions are true. (Hint: All the counterexamples can be constructed using the special case shown in the picture.)

*[2] Suppose $\mathcal{A}_1$ is a sigma-field on a set $\mathcal{X}_1$ and $\mathcal{A}_2$ is a sigma-field on a set $\mathcal{X}_1$. The product sigma-field $\mathcal{A}_1 \otimes \mathcal{A}_2$ is defined as $\sigma(\mathcal{E})$ where

$$\mathcal{E} = \{A_1 \times A_2 : A_1 \in \mathcal{A}_1 \text{ and } A_2 \in \mathcal{A}_2\}.$$

Suppose also that $\mathcal{A}_i = \sigma(\mathcal{E}_i)$ with $\mathcal{X}_i \in \mathcal{E}_i$, for each i. Follow these steps to show that $\mathcal{A}_1 \otimes \mathcal{A}_2$ is also generated by $\mathcal{F} = \{E_1 \times E_2 : E_1 \in \mathcal{E}_1 \text{ and } E_2 \in \mathcal{E}_2\}$. Note that $\mathcal{F} \subseteq \mathcal{E}$.

(i) Define $\mathcal{B} = \{B \in \mathcal{A}_1 : B \times \mathcal{X}_2 \in \sigma(\mathcal{F})\}$. Show that $\mathcal{B}$ is a sigma-field with $\mathcal{B} \supseteq \mathcal{E}_1$. Deduce that $A_1 \times \mathcal{X}_2 \in \sigma(\mathcal{F})$ for each $A_1 \in \mathcal{A}_1$.

(ii) Similarly (no need for proof), we have $\mathcal{X}_1 \times A_2 \in \sigma(\mathcal{F})$ for each $A_2 \in \mathcal{A}_2$. Show that $\mathcal{E} \subseteq \sigma(\mathcal{F})$.

(iii) Complete the argument.

*[3] Suppose $\mathcal{X}_1$ and $\mathcal{X}_2$ are metric spaces (or just topological spaces) equipped with their Borel sigma-fields: $\mathcal{B}(\mathcal{X}_i) = \sigma(\mathcal{G}_i)$, where $\mathcal{G}_i$ is the set of all open subsets of $\mathcal{X}_i$. By definition, the open subsets of $\mathcal{X}_1 \times \mathcal{X}_2$ are obtained by taking arbitrary unions of sets of the form $G_1 \times G_2$ with $G_i \in \mathcal{G}_i$.

(i) Show that $\mathcal{B}(\mathcal{X}_1 \times \mathcal{X}_2) \supseteq \mathcal{B}(\mathcal{X}_1) \otimes \mathcal{B}(\mathcal{X}_2)$.

(ii) Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. Hint: Think about the collection of all open rectangles of the form $(a_1, b_1) \times (a_2, b_2)$, with rational a_1, b_1, a_2, b_2.

[4] Suppose f and g are both real-valued $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable functions defined on a set $\mathcal{X}$ equipped with a sigma-field $\mathcal{A}$. Show that $x \mapsto f(x)g(x)$ is also $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable by figuring out the meaning of the following diagram.

$$\begin{array}{rcrcr}
x & \mapsto & T(x) & \mapsto & \psi(T(x)) \\
\mathcal{X}, \mathcal{A} & & \mathbb{R}^2, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) & & \\
 & & \mathbb{R}^2, \mathcal{B}(\mathbb{R}^2) & & \mathbb{R}, \mathcal{B}(\mathbb{R})
\end{array}$$

Here $T(x) = (f(x), g(x))$ and $\psi(u, v) = uv$. You may assume (without proof) that ψ is a continuous function.