Statistics 330b/600b, Math 330b spring 2018
Homework # 10
Due: Thursday 12 April

*[1] Suppose $\{\mathcal{F}_i : i \in \mathbb{N}_0\}$ is a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As before define $\mathcal{F}_\infty := \sigma(\cup_{i\in\mathbb{N}_0}\mathcal{F}_i)$. Suppose that τ is a stopping time for the filtration and that $X \in \mathcal{M}^+(\Omega, \mathcal{F})$. Show that X is $\mathcal{F}_\tau$-measurable if and only if the exists X_i in $\mathcal{M}^+(\mathcal{F}_i)$, for $0 \le i \le \infty$, such that $X(\omega) = \sum_{0\le i\le\infty} X_i(\omega)\mathbb{1}\{\omega : \tau(\omega) = i\}$.

*[2] Suppose $\{(X_i, \mathcal{F}_i) : i = 0, 1, \dots, n\}$ is a martingale with $\mathbb{P}X_i^2 < \infty$ for each i. Define $M := \max_{i\le n} |X_i|$.

(i) For each $t > 0$ define $\sigma(t) := \sigma_t(\omega)n \wedge \inf\{i : |X_i(\omega)| \ge t\}$. Show that $\sigma(t)$ is a stopping time for which

$$\{M \ge t\} = \{|X_{\sigma(t)}| \ge t\} \le |X_{\sigma(t)}|\{|X_{\sigma(t)}| \ge t\}/t.$$

(ii) Use the stronger form of the Stopping Time Lemma to deduce that

$$t\mathbb{P}\{M \ge t\} \le \mathbb{P}|X_n|\{M \ge t\} \qquad \text{for each } t > 0.$$

(iii) Integrate the last inequality with respect to t then invoke the Hölder inequality to conclude that $\mathbb{P}M^2 \le 4\mathbb{P}X_n^2$. Hint: If you plan on dividing by $\sqrt{\mathbb{P}M^2}$ you should explain why this quantity is neither zero nor infinite.

*[3] Suppose $\{(X_t, \mathcal{F}_t) : t \in \mathbb{N}_0\}$ is a martingale with $\Gamma^2 := \sup_{t\in\mathbb{N}_0} \mathbb{P}X_t^2 < \infty$. Define $\mathcal{F}_\infty = \sigma(\cup_{t\in\mathbb{N}_0}\mathcal{F}_t)$.

(i) Show that $\mathbb{P}(X_\ell - X_k)X_k = 0$ if $k < \ell$. Deduce that $\mathbb{P}X_t^2$ increases to Γ^2 as $t \to \infty$.

(ii) For ℓ and k in $\mathbb{N}_0$ with $\ell > k$ and $\delta > 0$ show that

$$\begin{aligned}\Gamma^2 \ge \mathbb{P}X_\ell^2 &= \mathbb{P}X_k^2 + \mathbb{P}(X_\ell - X_k)^2 \\ &\ge \Gamma^2 - \delta + \mathbb{P}(X_\ell - X_k)^2 \qquad \text{if } k \text{ is large enough.}\end{aligned}$$

Deduce that $\{X_t : t \in \mathbb{N}_0\}$ is a Cauchy sequence in $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, which converges in $\mathcal{L}^2$ to some $X_\infty \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$.

(iii) Show that $\|X_\infty\|_2 = \Gamma$.

(iv) For positive integers $k < \ell$, use Problem [2] to show that

$$\mathbb{P}\sup_{k\le t\le\ell} |X_t - X_k|^2 \le 4\mathbb{P}|X_\ell - X_k|^2.$$

Let ℓ tend to ∞ to deduce that, for each fixed k,

$$\mathbb{P}\sup_{t\ge k} |X_t - X_k|^2 \le 4\epsilon_k^2 := 4\mathbb{P}|X_\infty - X_k|^2.$$

(v) Define $Y_k := \sup_{t\ge k} |X_t - X_\infty|$. Show that

$$\mathbb{P}Y_k \le \left\|\sup_{t\ge k} |X_t - X_k|\right\|_2 + \|X_k - X_\infty\|_2 \le 3\epsilon_k.$$

(vi) Prove that $Y_k \downarrow 0$ a.e. $[\mathbb{P}]$. Deduce that $X_t \to X_\infty$ a.e. $[\mathbb{P}]$ as $t \to \infty$.

*(vii) Show that $X_t = \mathbb{P}(X_\infty \mid \mathcal{F}_t)$ a.e. $[\mathbb{P}]$.

[4] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G}$ be a sub-sigma-field of $\mathcal{F}$. Suppose $\{X_n : n \in \mathbb{N}\}$ is a sequence of random variables that converges to 0 a.e. $[\mathbb{P}]$. Suppose also that $|X_n| \le W \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for all n.

Define $Z_i = \mathbb{P}_{\mathcal{G}} X_i$ and $S_n = \sup_{i \ge n} |X_i|$ and $Y_n = \sup_{i \ge n} |Z_i|$. Prove that

$$\mathbb{P} Y_n \le \mathbb{P} S_n \to 0 \qquad \text{as } n \to \infty.$$

Deduce that $\mathbb{P}_{\mathcal{G}} X_n \to 0$ a.e. $[\mathbb{P}]$.

[5] (HARD) Suppose $\{Z_i : i \in \mathbb{N}_0\}$ is a sequence of random variables defined on Ω and $\mathcal{F}_n = \sigma(Z_0, Z_1, \dots, Z_n)$ for $n \in \mathbb{N}_0$. Let τ be a stopping time for that filtration.

(i) Explain why every $\mathcal{F}_n$-measurable (real valued) random variable Y can be written in the form $Y(\omega) = g_n(Z_0(\omega), \dots, Z_n(\omega))$ for some $\mathcal{B}(\mathbb{R}^{n+1})$-measurable function g_n. Hint: Lecture 3.

(ii) Define $X_i = Z_{\tau \wedge i}$ and $\mathcal{G} := \sigma(X_i : i \in \mathbb{N}_0)$. Prove that X_i is $\mathcal{F}_\tau$-measurable. Deduce that $\mathcal{G} \subseteq \mathcal{F}_\tau$. Hint: Split $\{X_i \in B\}\{\tau \le n\}$ into contributions from various sets $\{\tau = j\}$.

(iii) Prove that τ is $\mathcal{G}$-measurable. Hint: $\{\tau = 0\} = g_0(Z_0) = g_0(X_0)$ and $\{\tau = 1\} = g_1(Z_0, Z_1) = g_1(Z_0, Z_1)\{\tau \ge 1\} = g_1(X_0, X_1)\{\tau = 0\}^c$.

(iv) Show that $\mathcal{F}_\tau \subseteq \mathcal{G}$. Hint: If $F \in \mathcal{F}_\tau$ consider sets $F\{\tau = j\}$ for $j \in \mathbb{N}_0$.