Statistics 330b/600b, Math 330b spring 2018
Homework # 11
Due: Thursday 19 April

*[1] Suppose $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $Y \in \mathcal{L}^1(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathcal{G} = \sigma(\mathcal{E})$ is a sub-sigma-field of $\mathcal{F}$ and $\mathcal{E}$ is a generating class that is stable under pairwise intersections. Suppose $\mathbb{P}X = \mathbb{P}Y$ and $\mathbb{P}X\mathbb{1}_E = \mathbb{P}Y\mathbb{1}_E$ for each E in $\mathcal{E}$. Show that $Y = \mathbb{P}_{\mathcal{G}}X$ a.e. $[\mathbb{P}]$.

*[2] Suppose $\{(Z_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a submartingale and τ is a stopping time for the filtration. Define $X_n = Z_{n\wedge\tau}$. Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a submartingale. Hint: Consider the behavior of the difference $X_{n+1} - X_n$ on the sets $\{\tau \le n\}$ and $\{\tau \le n\}^c$.

*[3] In class I considered the gambler's ruin problem (UGMTP Exercise 6.23) where $\xi_1, \xi_2, \dots$ are independent with $\mathbb{P}\{\xi_i = +1\} = p$ and $\mathbb{P}\{\xi_i = -1\} = q = 1 - p$ for some p in $(0, 1)\backslash\{1/2\}$. I defined $W_n = a + \xi_1 + \dots + \xi_n$ and $Y_n = s^{W_n}$ for $s = q/p$. I argued that $\{(Y_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a martingale for the filtration $\mathcal{F}_n = \sigma\{\xi_1, \dots, \xi_n\}$. For the stopping time

$$\tau = \inf\{i : W_i \in \{0, a + b\}\},$$

I defined $M_n = Y_{n\wedge\tau}$. I asserted that $\mathbb{P}\{\tau < \infty\} = 1$. Prove this assertion, using only the fact that M_n converges a.e. $[\mathbb{P}]$ to a finite limit. Hint: Show that there exists a constant $c > 0$, which depends on s, a, and b, such that $|M_{n+1}(\omega) - M_n(\omega)| \ge c$ for all n if $\tau(\omega) = \infty$.

*[4] I have a collection of m different coins, the ith of which lands heads with probability θ_i when tossed. Let $\mathbb{Q}$ be a probability measure on $\{1, 2, \dots, m\}$ for which $\mathbb{Q}\{i\} = p_i$ for $i = 1, \dots, m$, where $p_i > 0$ for each i. I generate a sequence of random variables $X_1, X_2, \dots$ as follows. First generate an observation T from $\mathbb{Q}$. If $T = i$ then toss the ith coin repeatedly, recording $X_j = 1$ if the jth toss lands heads and $X_j = 0$ for tails.

(i) Show that $\mathbb{P}X_1 = \overline{\theta} := \sum_{i=1}^m p_i\theta_i$. Hint: Condition.

(ii) For each set $\{n_1, \dots, n_k\}$ of k distinct positive integers find $\mathbb{P}(\prod_{j=1}^k X_{n_j})$.

*(iii) Show that the sequence $(X_n, n \in \mathbb{N})$ is exchangeable (UGMTP Definition 6.49).

(iv) Show that $\operatorname{cov}(X_1, X_2) = 0$ if and only if $\theta_i = \overline{\theta}$ for every i.

*(v) Explain why $X_1, X_2, \dots$ are independent if and only if $\theta_i = \overline{\theta}$ for every i.

Remark. The ill-fated problem [4] has been causing more problems. You could interpret it the following way.

Suppose $\mathbb{P}_1, \dots, \mathbb{P}_m$ are probability measures on $(\Omega, \mathcal{F})$ and $X_1, X_2, \dots$ are measurable maps from Ω into $\{0, 1\}$. Under $\mathbb{P}_i$, the X_j's are independent, each with distribution $\text{Ber}(\theta_i)$. Define $\mathbb{P} = \sum_{i=1}^m p_i\mathbb{P}_i$.

Part (iii) effectively asks you to prove that, for each k and each permutation σ of $(1, 2, \dots, k)$ and all choices of $\alpha_1, \dots, \alpha_k$ from $\{0, 1\}$,

$$\mathbb{P}\{X_1 = \alpha_1, X_2 = \alpha_2, \dots, X_k\} = \mathbb{P}\{X_{\sigma(1)} = \alpha_1, X_{\sigma(2)} = \alpha_2, \dots, X_{\sigma(k)}\}.$$

Hint: Note that $\mathbb{1}\{X_j = 1\} = X_j$ and $\mathbb{1}\{X_j = 0\} = 1 - X_j$. That leads to simplifications such as

$$\begin{aligned}\mathbb{1}\{X_1 = 1, X_2 = 0, X_3 = 1, X_4 = 0\} &= X_1(1 - X_2)X_3(1 - X_4)\\ &= X_1X_3 - \dots + X_1X_2X_3X_4.\end{aligned}$$