S&DS 400b/600b, Math 330b, spring 2018
Homework # 12
Due: Thursday 26 April

*[1] Let A be a subset of a metric space $(\mathcal{X}, d)$. For each x in $\mathcal{X}$ define

$$d(x, A) := \inf\{d(x, y) : y \in A\}.$$

(i) For each pair of points x_1 and x_2 in $\mathcal{X}$, show that

$$|d(x_1, A) - d(x_2, A)| \le d(x_1, x_2).$$

(ii) Show that

$$\{x : d(x, A) = 0\} = \bar{A} := \text{closure of } A,$$
$$\{x : d(x, A^c) > 0\} = \mathring{A} := \text{interior of } A,$$
$$\{x : d(x, A) = 0 = d(x, A^c)\} = \bar{A}\backslash\mathring{A} = \partial A = \text{boundary of } A.$$

Hint: If $d(x, A) = 0$, there exists points $x_n \in A$ with $d(x, x_n) \to 0$.

[2] Suppose f_1 and f_2 are real valued functions on a metric space $(\mathcal{X}, d)$.

(i) If both functions are Lipschitz, with $\|f_i\|_{\text{Lip}} = \alpha_i$, and $f(x) := \max(f_1(x), f_2(x))$ for each x, show that $\|f\|_{\text{Lip}} \le \max(\alpha_1, a_2)$. Hint: If $d(x, y) = \delta$, show that

$$f(x) \le \max(f_1(y) + \alpha_1\delta, f_2(y) + \alpha_2\delta).$$

(ii) If both functions belong to $\text{BL}(\mathcal{X})$, with $\|f_i\|_{\text{BL}} = \beta_i$, prove that $\|f_1 f_2\|_{\text{BL}} \le 2\beta_1\beta_2$.

*[3] Suppose $P, P_1, P_2, \ldots$ are probability measures on the Borel sigma-field $\mathcal{B}(\mathcal{X})$ of a metric space $(\mathcal{X}, d)$ for which

$$P_n B \to PB \qquad \text{for each } B \in \mathcal{B}(\mathcal{X}) \text{ for which } P(\partial B) = 0.$$

Let f be a bounded $\mathcal{B}(\mathcal{X})$-measurable function from $\mathcal{X}$ to $[0, 1]$ that is continuous at all points of $\mathcal{X}$ except those of a P-negligible set $\mathcal{N}$.

(i) For each t in $[0, 1]$, show that the boundary of the set $B_t := \{f \ge t\}$ is contained in $\mathcal{N} \cup \{f = t\}$. Hint: If $x \in \partial B_t\backslash\mathcal{N}$ there exist sequences $x_n \to x$ and $y_n \to x$ with $x_n \in B_t$ and $y_n \in B_t^c$.

(ii) Deduce that $P(\partial B_t) > 0$ for at most countably many values of t.

(iii) Show that $P_n f = \int_0^1 P_n\{x : f(x) \ge t\}\, dt \to Pf$.

*[4] Suppose K is a continuously differentiable function on $\mathbb{R}$ which is zero outside some bounded interval and for which $\int_{\mathbb{R}} K(x)\, dx = 1$. For a given bounded measurable function f on $\mathbb{R}$ and each $\sigma > 0$ define

$$f_\sigma(x) := \int_{\mathbb{R}} f(x + \sigma y)K(y)\, dy = \frac{1}{\sigma}\int_{\mathbb{R}} f(z)K\left(\frac{z - x}{\sigma}\right) dz.$$

(i) If $f \in \text{BL}(\mathcal{X})$ with $\|f\|_{\text{BL}} = C$ show that $|f_\sigma(x) - f(x)| \le C\sigma\int_{\mathbb{R}} |yK(y)|\, dy$ for every x.

(ii) Show (rigorously) that f_σ is differentiable with $\sigma f_\sigma'(x) = -\int f(x + \sigma y)K'(y)\, dy$.

(iii) Explain why f_σ belongs to $\mathcal{C}^\infty(\mathbb{R})$ (= the set of all bounded real functions with bounded derivatives of all orders) if $K \in \mathcal{C}^\infty(\mathbb{R})$.

[5] (Needed if you are interested in Fourier transforms, which use integrals of complex valued functions.) Suppose $f_1, f_2 \in \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Show that

$$\mu|f_1(x) + if_2(x)| \ge |\mu(f_1) + i\mu(f_2)|.$$

Hint: Define $F = \sqrt{f_1^2 + f_2^2}$ and $C = \mu F$. Show $C < \infty$. If $C \ne 0$ define P to be the probability measure with density F/C with respect to μ. Define $g_j = (f_j/F)\mathbb{1}\{F > 0\}$. What does Pg_j equal?