Statistics 330b/600b, Math 330b spring 2018
Homework # 2
Due: Thursday 1 February

Please attempt at least the starred problems. Please explain your reasoning. Keep looking at latex.pdf.

*[1] A set $\mathcal{E}$ of subsets of $\mathcal{X}$ is called a field if: it contains $\emptyset$ and is stable under complements, finite unions, and finite intersections. Suppose μ is a finite measure (that is, $\mu\mathcal{X} < \infty$) on a sigma-field $\mathcal{A} = \sigma(\mathcal{E})$, where $\mathcal{E}$ is a field of subsets of $\mathcal{X}$. Show that every A in $\mathcal{A}$ has the following property:

$$(\star)\text{: for each } \epsilon > 0 \text{ there exists an } E \in \mathcal{E} \text{ for which } \mu(A\Delta E) < \epsilon.$$

Remember that $A\Delta E := (A \cap E^c) \cup (A^c \cap E)$. Follow these steps.

(i) Explain why $(\star)$ is equivalent to:

$$(\star\star)\text{: for each } \epsilon > 0 \text{ there exists an } E \in \mathcal{E} \text{ for which } \widetilde{\mu}|\mathbb{1}_A - \mathbb{1}_E| < \epsilon.$$

(ii) Define $\mathcal{A}_0 := \{A \in \mathcal{A} : A \text{ has property } (\star\star)\}$. Show that $\mathcal{A}_0 \supseteq \mathcal{E}$.

(iii) For all A and E show that $|\mathbb{1}_{A^c} - \mathbb{1}_{E^c}| = |\mathbb{1}_A - \mathbb{1}_E|$.

(iv) Suppose $\{A_i : i \in \mathbb{N}\} \subset \mathcal{A}$ and $\{E_i : i \in \mathbb{N}\} \subset \mathcal{E}$. Define $A = \cup_{i\in\mathbb{N}} A_i$ and $D = \cup_{i\in\mathbb{N}} E_i$ and $D_n = \cup_{i\le n} E_i$. Show that

$$|\mathbb{1}_A - \mathbb{1}_D| \le \sum\nolimits_{i\in\mathbb{N}} |\mathbb{1}_{A_i} - \mathbb{1}_{E_i}|$$

$$|\mathbb{1}_D - \mathbb{1}_{D_n}| \to 0 \qquad \text{pointwise as } n \to \infty.$$

(v) Deduce that $\mathcal{A}_0$ is a sigma-field then complete the argument.

(vi) (optional extra) Show that the set $\mathcal{H}$ of all finite linear combinations of the form $\sum_i \alpha_i \mathbb{1}_{E_i}$, with $\alpha \in \mathbb{R}$ and $E_i \in \mathcal{E}$, is dense (under $\|\cdot\|_1$) in $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$.

*[2] Suppose $f_1, \dots, f_k \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\theta_1, \dots, \theta_k$ are strictly positive numbers that sum to one. Let μ be a measure on $\mathcal{A}$. Show that

$$\mu \prod\nolimits_{i\le k} f_i^{\theta_i} \le \prod\nolimits_{i\le k} (\mu f_i)^{\theta_i} \qquad \text{<1>}$$

by following these steps. You may use the inequality

$$\prod\nolimits_{i=1}^k a_i^{\theta_i} \le \sum\nolimits_{i=1}^k \theta_i a_i \qquad \text{for all } a_i \in [0, \infty), \qquad \text{<2>}$$

which, as explained in class, is a simple consequence of the concavity of the log function.

(i) Explain why inequality <1> is trivially true if $\mu f_i = 0$ for at least one i.

(ii) Explain why inequality <1> is trivially true if $\mu f_i > 0$ for all i and $\mu f_i = +\infty$ for at least one i.

(iii) Explain why there is no loss of generality in assuming that $\mu f_i = 1$ for each i and $f_i(x) < \infty$ for each x and i.

(iv) Complete the proof by considering the inequality <2> with $a_i = f_i(x)$.

> **Remark.** Textbooks often contain the the special case where $k = 2$ and $\theta_1 = 1/p$ and $\theta_2 = 1/q$ and $f_1 = |g_1|^p$ and $f_2 = |g_2|^q$, with the assertion that $|\mu(g_1 g_2)| \le \mu|g_1 g_2| \le (\mu|g_1|^p)^{1/p} (\mu|g_2|^q)^{1/q}$.

[3] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and f is an $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable function from $\mathcal{X}$ into $\mathbb{R}$. Suppose the function $g(x,\theta) = e^{\theta f(x)}$ belongs to $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ for each fixed real number θ in a (non-degenerate) interval $[-\delta, \delta]$. Define $M(\theta) = \widetilde{\mu} e^{\theta f(x)}$ and $L(\theta) = \log M(\theta)$, for $-\delta \le \theta \le \delta$.

(i) Show that M is differentiable in the open interval $(-\delta, \delta)$, with derivative

$$M'(\theta) = \widetilde{\mu}(f e^{\theta f}).$$

(ii) Show that M' is differentiable in the open interval $(-\delta, \delta)$, with derivative

$$M''(\theta) = \widetilde{\mu}(f^2 e^{\theta f}).$$

(iii) For each θ in $[-\delta, \delta]$ define P_θ to be the (probability) measure defined to have density $e^{\theta f(x)}/M(\theta)$ with respect to μ. On the open interval $(-\delta, \delta)$ show that

$$L'(\theta) = P_\theta f$$

and

$$L''(\theta) = P_\theta(f^2) - (P_\theta f)^2 \ge 0$$

Hint: Think variance.

(iv) Deduce that L is a convex function, at least on the open interval $(-\delta, \delta)$.