Statistics 330b/600b, Math 330b spring 2018
Homework # 3
Due: Thursday 8 February

Please attempt at least the starred problems. [cf. $(\forall x \in \emptyset)(x$ is green).] Please explain your reasoning. Keep looking at latex.pdf.

[1] Mimic the proof in the "Convergent subsequences" handout to prove that $\mathcal{L}^p(\mathcal{X}, \mathcal{A}, \mu)$ is complete for $1 < p < \infty$. If you are very brave you could also prove completeness of $\mathcal{L}^\Psi(\mathcal{X}, \mathcal{A}, \mu)$ for each Orlicz space, simplifying the proof of UGMTP Problem 2.23. If you are less brave, prove completeness just for $\mathcal{L}^2(\mathcal{X}, \mathcal{A}, \mu)$.

[2] Let Ψ be a convex, increasing function for which $\Psi(0) = 0$ and $\Psi(x) \to \infty$ as $x \to \infty$. (For example, $\Psi(x)$ could equal x^p for some fixed $p \geq 1$, or $\exp(x) - 1$ or $\exp(x^2) - 1$.) Define $\mathcal{L}^\Psi(\mathcal{X}, \mathcal{A}, \mu)$ to be the set of all real-valued measurable functions on $\mathcal{X}$ for which $\mu\Psi(|f|/c_0) < \infty$ for some positive real c_0. Define $\|f\|_\Psi := \inf\{c > 0 : \mu\Psi(|f|/c) \leq 1\}$, with the convention that the infimum of an empty set equals $+\infty$. For each f, g in $\mathcal{L}^\Psi$ and each real t prove the following assertions.

(i) $\|f\|_\Psi < \infty$. Hint: Apply Dominated Convergence to $\mu\Psi(|f|/c)$.

(ii) $f + g \in \mathcal{L}^\Psi$ and the triangle inequality holds: $\|f + g\|_\psi \leq \|f\|_\psi + \|g\|_\psi$. Hint: If $c > \|f\|_\psi$ and $d > \|g\|_\Psi$, deduce that

$$\Psi\left(\frac{|f+g|}{c+d}\right) \leq \frac{c}{c+d}\Psi\left(\frac{|f|}{c}\right) + \frac{d}{c+d}\Psi\left(\frac{|g|}{d}\right),$$

by convexity of Ψ.

(iii) $tf \in \mathcal{L}^\Psi$ and $\|tf\|_\psi = |t|\ \|f\|_\psi$.

Remark. $\|\cdot\|_\psi$ is called an Orlicz "norm"—to make it a true norm one should work with equivalence classes of functions equal μ almost everywhere. The L^p norms correspond to the special case $\Psi(x) = x^p$, for some $p \geq 1$.