Statistics 330b/600b, Math 330b spring 2018
Homework # 4
Due: Thursday 15 February

Please attempt at least the starred problems. Please explain your reasoning. Keep looking at latex.pdf. Please try to avoid strings of ∴ (???) ∵ (???) ∴ (???) ∵ (???) . . . . They really can be hard to decipher.

*[1] Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Suppose also that two functions X_1, X_2 from $\mathcal{M}^+(\Omega, \mathcal{F})$ have the property that $\mathbb{P}(X_1 \mathbb{1}_F) = \mathbb{P}(X_2 \mathbb{1}_F)$ for all F in $\mathcal{F}$. Prove that $X_1 = X_2$ a.e. $[\mathbb{P}]$. Hint: Consider sets

$$F_{r,s} = \{\omega \in \Omega : X_1(\omega) < r < s < X_2(\omega)\}.$$

*[2] Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of (real-valued) random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose the sequence is Cauchy in probability, that is, for each $\epsilon > 0$ and $\delta > 0$ there exists an $n_{\epsilon,\delta}$ for which

$$\mathbb{P}\{\omega \in \Omega : |X_n(\omega) - X_m(\omega)| > \delta\} < \epsilon \qquad \text{whenever } \min(m,n) \geq n_{\epsilon,\delta}.$$

From the subsequence.pdf handout you know there exists a random variable X and a subsequence $\{n(k) : k \in \mathbb{N}\}$ for which

$$X_{n(k)}(\omega) \to X(\omega) \qquad \text{a.e. } [\mathbb{P}] \text{ as } k \to \infty.$$

(i) Explain why $\mathbb{1}\{\omega : |X_{n(k)}(\omega) - X(\omega)| > \delta\} \to 0$ a.e. $[\mathbb{P}]$ for each $\delta > 0$. Deduce that $X_{n(k)}$ converges in probability to X.

(ii) Explain why

$$\mathbb{1}\{|X_n - X| > 2\delta\} \leq \mathbb{1}\{|X_n - X_{n(k)}| > \delta\} + \mathbb{1}\{|X_{n(k)} - X| > \delta\}.$$

Deduce that X_n converges in probability to X as $n \to \infty$.

*[3] Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{A_i : i \in \mathbb{N}\} \subset \mathcal{F}$. Define $S_n = \mathbb{1}_{A_1} + \cdots + \mathbb{1}_{A_n}$ and $S_\infty = \sum_{i \in \mathbb{N}} \mathbb{1}_{A_i}$. In class I showed that $\mathbb{P}\{S_\infty \geq 1\} = 1$ if $\|S_n\|_1 \to \infty$ and $\|S_n\|_2 / \|S_n\|_1 \to 1$ as $n \to \infty$.

(i) Show that $\mathbb{P}\{S_\infty - S_m \geq 1\} = 1$ for each $m \in \mathbb{N}$.

(ii) Deduce that $\mathbb{P}\{\omega \in A_i \text{ for infinitely many } i\} = 1$.

(iii) Suppose $\{B_i : i \in \mathbb{N}\} \subset \mathcal{F}$ satisfies $\sum_{i \in \mathbb{N}} \mathbb{P}B_i = \infty$ and $\mathbb{P}(B_i B_j) = (\mathbb{P}B_i)(\mathbb{P}B_j)$ for all $i \neq j$. Show that $\mathbb{P}\{\omega \in B_i \text{ for infinitely many } i\} = 1$. *Please use (ii). I am not interested in seeing the standard textbook proof for the harder direction of Borel-Cantelli.*

[4] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and μ is a measure on $\mathcal{A}$. Write $\mathcal{N}_\mu$ for $\{N \in \mathcal{A} : \mu N = 0\}$. Define

$$\mathcal{A}_\mu := \{B \subseteq \mathcal{X} : \exists A \in \mathcal{A},\, N \in \mathcal{N}_\mu \text{ such that } |\mathbb{1}_B - \mathbb{1}_A| \leq \mathbb{1}_N\}.$$

(i) Show that $\mathcal{A}_\mu$ is a sigma-field.

(ii) If $|\mathbb{1}_B - \mathbb{1}_{A_i}| \leq \mathbb{1}_{N_i}$ for $i = 1, 2$, with $A_i \in \mathcal{A}$ and $N_i \in \mathcal{N}_\mu$, show that $\mu A_1 = \mu A_2$.

(iii) If $|\mathbb{1}_B - \mathbb{1}_A| \leq \mathbb{1}_N$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}_\mu$ define $\nu B = \mu A$. Show that ν is a well defined countably additive measure on $\mathcal{A}_\mu$ whose restriction to $\mathcal{A}$ equals μ.

[5] Define $x_n = k_n/2^n$, where k_n is the largest integer for which $(k_n/2^n)^2 \leq 3$. Prove that $\{x_n : n \in \mathbb{N}\}$ is a Cauchy sequence of real numbers.