Statistics 330b/600b, Math 330b spring 2018
Homework # 6
Due: Thursday 1 March

Please attempt ....

*[1] Suppose X and Y are independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. That is,

$$\mathbb{P}\{X \in A, Y \in B\} = \mathbb{P}\{X \in A\}\mathbb{P}\{Y \in B\} \qquad \text{for all } A, B \in \mathcal{B}(\mathbb{R}).$$

Suppose also that each random variable has a $\text{Bin}(2, p)$ distribution, for some fixed p in $(0, 1)$.

(i) Define $T : \Omega \to \mathbb{R}^2$ by $T(\omega) = \big(X(\omega), Y(\omega)\big)$. Find the distribution, P, of T under $\mathbb{P}$. (That is, P is the image of $\mathbb{P}$ under T). Hint: The distribution should be a probability measure concentrated on nine points in $\mathbb{R}^2$.

(ii) Define $\psi : \mathbb{R}^2 \to \mathbb{R}$ by $\psi(x, y) = x + y$. Find the distribution, μ, of ψ under P.

(iii) Define $S : \Omega \to \mathbb{R}$ by $S(\omega) = X(\omega) + Y(\omega)$. Find the distribution, Q, of S under $\mathbb{P}$.

(iv) Calculate $\mathbb{P}(X + Y)$ in at least two different ways.

(v) Could you find $\mathbb{P}(X + Y)$ without knowing that X and Y are independent?

(vi) Find the distribution of XY. Find $\mathbb{P}(XY)$.

(vii) Could you find $\mathbb{P}(XY)$ without knowing that X and Y are independent?

[2] Suppose $X_1, X_2, \dots$ are independent random variables for which $\mathbb{P}X_i = 0$ for every i and $\sup_i \mathbb{P}X_i^6 \le M < \infty$. Prove that

$$\mathbb{P}\left(X_1 + \dots + X_n\right)^6 \le Cn^3 \qquad \text{for each } n,$$

where C is a finite constant that depends on M.

[3] Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space for which $\mathcal{F}$ is independent of itself. Prove that $\mathbb{P}F \in \{0, 1\}$ for every F in $\mathcal{F}$.

*[4] Suppose X is a random variable that is independent of itself. Prove that there exists a real number c for which $\mathbb{P}\{X = c\} = 1$. Hint: Consider the infimum of the set $\{t \in \mathbb{R} : \mathbb{P}\{X \le t\} = 1\}$.

*[5] (A version of Bennett's inequality) Suppose $X_1, X_2, \dots, X_n$ are independent random variables for which $\mathbb{P}X_i = 0$ and $\mathbb{P}X_i^2 \le \sigma^2 < \infty$ and $\mathbb{P}\{X_i \le 1\} = 1$ for every i. In what follows you may use, without proof, the fact that the function

$$\Delta(t) = \frac{e^t - 1 - t}{t^2/2}\mathbb{1}\{t \ne 0\} + \mathbb{1}\{t = 0\}$$

is increasing over the whole real line. [Of course you are welcome to offer a proof.]

(i) For each i and each $\theta > 0$, prove that

$$\mathbb{P}\exp\left(\theta X_i\right) \le 1 + \tfrac{1}{2}\theta^2\sigma_i^2\Delta(\theta) \le \exp\left(\sigma_i^2(e^\theta - 1 - \theta)\right)$$

(ii) Define $W = \sum_{i \le n} \sigma_i^2$. For each $x \ge 0$ show that

$$\begin{aligned}\mathbb{P}\{\textstyle\sum_{i\le n} X_i \ge x\} &\le \inf\nolimits_{\theta \ge 0} \exp\left(-\theta(x + W) + W(e^\theta - 1)\right) \\ &\approx \exp\left(-x^2/(2W)\right) \qquad \text{if } x/W \approx 0.\end{aligned}$$