Statistics 330b/600b, Math 330b spring 2018
Homework # 7
Due: Thursday 8 March

*[1] Suppose $(\mathcal{X}, \mathcal{A}.\mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ are measure space with both μ and ν sigma-finite. Suppose also that $f \in \mathcal{L}^1(\mathcal{X} \times \mathcal{Y}, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$. Define

$$\mathcal{X}_0 = \{x \in \mathcal{X} : \nu^y |f(x,y)| < \infty\} \quad \text{AND} \quad \mathcal{Y}_0 = \{y \in \mathcal{Y} : \mu^x |f(x,y)| < \infty\}.$$

Define $g(x,y) := f(x,y)\mathbb{1}\{x \in \mathcal{X}_0, y \in \mathcal{Y}_0\}$.

(i) Explain why g is $\mathcal{A} \otimes \mathcal{B}$-measurable.

(ii) Show that $g \in \mathcal{L}^1(\mathcal{X} \times \mathcal{Y}, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ and

$$\mu \otimes \nu f = \mu \otimes \nu g = \mu^x[\nu^y g(x,y)] = \nu^y[\mu^x g(x,y)]$$

in the sense that the last two expressions are well defined and are equal to the first two expressions.

(iii) Why might the same assertion with g everywhere replaced by f be suspect?

*[2] Let P be the probability measure on $\mathcal{B}(\mathbb{R})$ with density $p(x) = e^{-x}\mathbb{1}\{x > 0\}$ with respect to Lebesgue measure. Suppose X and Y are independent random variables, each with distribution P. Show that the distribution of $X - Y$ has density $\gamma(z) = \frac{1}{2}\exp(-|z|)$ with respect to Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

*[3] Suppose X and Y are independent real-valued random variables with

$$\mathbb{P}\{X = t\}\mathbb{P}\{Y = t\} = 0 \qquad \text{for each } t \in \mathbb{R}.$$

Show that $\mathbb{P}\{X = Y\} = 0$.
Hint: First show that the set $S := \{t \in \mathbb{R} : \mathbb{P}\{Y = t\} > 0\}$ is at worst countably infinite. Consider the function $g(x) = \sum_{t \in S} \mathbb{1}\{x = t\}\mathbb{P}\{Y = t\}$.

[4] For real-valued random variables X_1 and X_2, define

$$f(\omega, s) := \mathbb{1}\{X_1(\omega) > s\} + \mathbb{1}\{X_2(\omega) > s\} - 2\mathbb{1}\{X_1(\omega) > s, X_2(\omega) > s\}.$$

Show that $\int_{\mathbb{R}} f(\omega, s)\, ds = |X_1(\omega) - X_2(\omega)|$. Then complete the argument begun in class to show that $\mathbb{P}|X_1 - X_2|$ is minimized over all integrable random variables with $X_i \sim P_i$ if the X_i's are coupled using the quantile transformation.

[5] For each fixed $p > 0$ define $B_{p,n} := \{x \in \mathbb{R}^n : g_p(x) \le 1\}$ where $g_p(x) := \sum_{i \le n} |x_i|^p$. Follow these steps to show that

$$V_{p,n} := \text{vol}(B_{p,n}) = \frac{(2\Gamma(1 + 1/p))^n}{\Gamma(1 + n/p)}.$$

Remark. Recall that the Gamma function is defined for $\alpha > 0$ by $\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}dt$. Integration by parts shows that $\Gamma(1 + \alpha) = \alpha\Gamma(\alpha)$ for each $\alpha > 0$.

(i) For $x \in \mathbb{R}^n$ Show that

$$I_p := \int_{\mathbb{R}^n} \exp\left(-g_p(x)\right)\, dx = \left(2\int_0^\infty \exp(-t^p)\, dt\right)^n = \left(\frac{2}{p}\Gamma(1/p)\right)^n.$$

(ii) Show that

$$\exp\left(-g_p(x)\right) = \int_0^\infty \{t \ge g_p(x)\}e^t\, dt = \int_0^\infty \{t^{-1/p}x \in B_{p,n}\}e^{-t}\, dt.$$

(iii) Deduce that $I_p = V_p \int_0^\infty t^{n/p}e^{-t}dt$.

(iv) Then what?