Statistics 330b/600b, Math 330b spring 2018
Homework # 8
Due: Thursday 29 March

*[1] Suppose $(\mathcal{Y}, \mathcal{B}, \mathbb{Q})$ is a probability space and $g_1, g_2 \in \mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ have the property that

$$\mathbb{Q}g_1(y)\mathbb{1}\{y \in B\} \leq \mathbb{Q}g_1(y)\mathbb{1}\{y \in B\} \qquad \text{for all } B \text{ in } \mathcal{B}.$$

Prove that $g_1(y) \leq g_2(y)$ a.e. $[\mathbb{Q}]$.
Hint: Consider sets of the form $\{g_1(y) \geq r > s \geq g_2(y)\}$ for constants r and s.

[2] Let $\mathcal{K}_0$ be an essentially closed, convex subset of $\mathcal{L}^2 := \mathcal{L}^2(\mathcal{X}, \mathcal{A}, \mu)$. For a fixed f in $\mathcal{H}\backslash\mathcal{K}_0$ define $\delta := \inf\{\|f - h\|_2 : h \in \mathcal{K}_0\}$. Show that:

(i) There is an f_0 (unique up to μ-equivalence) in $\mathcal{K}_0$ for which $\|f - f_0\|_2 = \delta$.

(ii) $\mathcal{K}_0$ is contained in the half-space $\{h \in \mathcal{L}^2 : \langle h - f_0, f - f_0 \rangle \leq 0\}$.

You do do need to repeat steps that are identical to those in the Hilbert handout. You might find it helpful to draw a picture of $t \mapsto \|f - (1-t)f_0 - th\|_2^2 - \delta^2$ *for a fixed* h *in* $\mathcal{K}_0$.

*[3] Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and $\mathcal{B} = \sigma(\mathcal{E})$ is a sigma-field on a set $\mathcal{Y}$. Suppose also that $\mathcal{E}$ is a countable field (stable under complements, pairwise unions, and pairwise intersection) that separates the points of $\mathcal{Y}$: if $y_1 \neq y_2$ there exists a set $E \in \mathcal{E}$ for which $\mathbb{1}_E(y_1) \neq \mathbb{1}_E(y_2)$. For a given $\mathcal{A}\backslash\mathcal{B}$-measurable map T from $\mathcal{X}$ into $\mathcal{Y}$, define $G := \{(x, y) \in \mathcal{X} \times \mathcal{Y} : y = Tx\}$.

(i) Define $H := \cup_{E \in \mathcal{E}}(T^{-1}(E^c)) \times E$. Show that $H \subseteq G^c$.

(ii) If $(x, y) \in G^c$ show that there exists an $E \in \mathcal{E}$ for which $\mathbb{1}_E(Tx) \neq \mathbb{1}_E(y)$.

(iii) Deduce that $G = H^c \in \mathcal{A} \otimes \mathcal{B}$.

(iv) Suppose $\mathbb{P}$ is a probability measure on $\mathcal{A}$ whose image under T is a probability measure $\mathbb{Q}$ on $\mathcal{B}$. Define $\psi(x) = (x, Tx)$. Let γ be the image (on $\mathcal{A} \otimes \mathcal{B}$) of $\mathbb{P}$ under ψ. Finally, suppose $\mathcal{K} = \{K_y : y \in \mathcal{Y}\}$ is a set of probability measures on $\mathcal{A}$ for which $\gamma f = \mathbb{Q}^y K_y^x f(x, y)$ for each f in $\mathcal{M}^+(\mathcal{X} \times \mathcal{Y}, \mathcal{A} \otimes \mathcal{B})$. Show that

$$K_y\{x \in \mathcal{X} : Tx \neq y\} = 0 \qquad \text{a.e. } [\mathbb{Q}].$$

[4] Suppose $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \mu)$ are both measure spaces and T is an $\mathcal{A}\backslash\mathcal{B}$-measurable map from $\mathcal{X}$ to $\mathcal{Y}$. Suppose also that $\Lambda = \{\lambda_y : y \in \mathcal{Y}\}$ is a set of measures on $\mathcal{A}$ for which

$$\lambda f(x) = \mu^y \lambda_y^x f(x) \qquad \text{for each } f \text{ in } \mathcal{M}^+(\mathcal{X}, \mathcal{A}).$$

and $\lambda_y\{x : Tx \neq y\} = 0$ a.e. $[\mu]$.

> **Remark.** It is implicit that $y \mapsto \lambda_y^x f(x)$ should be $\mathcal{B}$-measurable. UGMTP Appendix F gives conditions under which such a decomposition exists.

Suppose $\mathbb{P}$ is a probability measure on $\mathcal{A}$ that has density $p(x)$ with respect to λ. Let $\mathbb{Q}$ be the image of $\mathbb{P}$ under T.

(i) Show that $\mathbb{Q}$ has density $q(y) := \lambda_y^x p(x)$ with respect to μ and $\mathbb{Q}\{y : q(y) = 0\} = 0$.

(ii) Define $\mathbb{P}_y$ to be the measure on $\mathcal{A}$ that has density $p(x \mid y) = \mathbb{1}\{q(y) > 0\}p(x)/q(y)$ with respect to λ_y. Show that $y \mapsto \mathbb{P}_y f(x)$ is $\mathcal{B}$-measurable for each f in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$.

(iii) For each f in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$ show that

$$\mathbb{Q}^y \mathbb{P}_y^x f(x) = \mathbb{P}^x \mathbb{1}\{q(Tx) > 0\} f(x) = \mathbb{P} f(x).$$