Statistics 330b/600b, Math 330b spring 2018
Homework # 9
Due: Thursday 5 April

*[1] Suppose $(\mathcal{X},\mathcal{A},\mathbb{P})$ is a probability space and $T:\mathcal{X}\to\mathcal{Y}=\{1,2,\dots,k\}$ is a map for which $B_i=\{x:Tx=i\}$ is $\mathcal{A}$-measurable, for each i. Let $\mathcal{B}$ denote the sigma-field generated by the sets $B_1,\dots,B_k$.

(i) Show that each g in $\mathcal{M}^+(\mathcal{X},\mathcal{B})$ must be of the form $\sum_{i=1}^k \alpha_i \mathbb{1}\{x\in B_i\}$, for constants α_i with $0\le\alpha_i\le\infty$. Hint: Why could there not exist points x_1 and x_2 in some B_i for which $g(x_1)\ne g(x_2)$?

(ii) If $\mathbb{Q}\{i\}>0$ define $K_i f=\mathbb{P}(f(x)\mathbb{1}\{x\in B_i\})/\mathbb{P}B_i$ for $f\in\mathcal{M}^+(\mathcal{X},\mathcal{A})$. If $\mathbb{Q}\{i\}=0$ let K_i be the zero measure on $\mathcal{A}$, that is, $K_i f=0$ for $f\in\mathcal{M}^+(\mathcal{X},\mathcal{A})$. Show that $\mathbb{K}=\{K_i:1\le i\le k\}$ provides a conditional distribution for $\mathbb{P}$ given T.

(iii) Find $\mathbb{P}_{\mathcal{B}} f$ for $f\in\mathcal{M}^+(\mathcal{X},\mathcal{A})$.

*[2] (Neyman factorization theorem cf. UGMTP Example 5.31) Suppose $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space and $\{\mathbb{P}_\theta:\theta\in\Theta\}$ is a set of probability measures on $\mathcal{F}$ with each $\mathbb{P}_\theta$ absolutely continuous with respect to $\mathbb{P}$. Suppose also that $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ and that $\mathbb{P}_\theta$ has density

$$p(\omega,\theta)=g_\theta(\omega)h(\omega)\qquad\text{with } g_\theta\in\mathcal{M}^+(\Omega,\mathcal{G})\text{ for each }\theta$$

for a fixed $h\in\mathcal{M}^+(\Omega,\mathcal{F})$ that doesn't depend on θ. That is, $\mathbb{P}_\theta f=\mathbb{P}\left(g_\theta(\omega)h(\omega)f(\omega)\right)$ for each f in $\mathcal{M}^+(\Omega,\mathcal{F})$.

Let H be a function (unique up to $\mathbb{P}$-equivalence) in $\mathcal{M}^+(\Omega,\mathcal{G})$ for which

$$\mathbb{P}g(\omega)h(\omega)=\mathbb{P}g(\omega)H(\omega)\qquad\text{for each } g \text{ in } \mathcal{M}^+(\Omega,\mathcal{G}).$$

(That is H is a possible choice for $\mathbb{P}(h\mid\mathcal{G})$.)

(i) Show that $\mathbb{P}_\theta\{H=0\}=0$ for each θ.

(ii) Show that $\mathbb{P}_\theta\{H=\infty\}=0$ for each θ. Hint: Start by showing that

$$1\ge n\mathbb{P}g_\theta\mathbb{1}\{H(\omega)=\infty\}\qquad\text{for each } n\in\mathbb{N}.$$

What does that tell you about $g_\theta(\omega)\mathbb{1}\{H(\omega)=\infty\}$?

(iii) For each X in $\mathcal{M}^+(\Omega,\mathcal{F})$ and some fixed choice of γ of $\mathbb{P}(Xh\mid\mathcal{G})$ define

$$Y(\omega)=\frac{\gamma(\omega)}{H(\omega)}\mathbb{1}\{0<H<\infty\}.$$

Show that $\mathbb{P}_\theta(X\mid\mathcal{G})=Y$ a.e $[\mathbb{P}_\theta]$ for every θ.

[3] Give a complete proof of the Radon-Nikodym theorem (Theorem $\langle 21\rangle$ in the Hilbert handout). That is, turn the sketch into a real proof.