S&DS 400/600, spring 2018
Homework #10 solutions

*[1] *Suppose $\{\mathcal{F}_i : i \in \mathbb{N}_0\}$ is a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As before define $\mathcal{F}_\infty := \sigma(\cup_{i\in\mathbb{N}_0}\mathcal{F}_i)$. Suppose that τ is a stopping time for the filtration and that $X \in \mathcal{M}^+(\Omega, \mathcal{F})$. Show that X is $\mathcal{F}_\tau$-measurable if and only if there exists X_i in $\mathcal{M}^+(\mathcal{F}_i)$, for $0 \le i \le \infty$, such that $X(\omega) = \sum_{0\le i\le\infty} X_i(\omega)\mathbb{1}\{\omega : \tau(\omega) = i\}$.*

Solution: *Remember that the set $\{\tau = i\}$ is both $\mathcal{F}_i$-measurable and $\mathcal{F}_\tau$-measurable.*

If X is $\mathcal{F}_\tau$-measurable define $X_i = X\mathbb{1}\{\tau = i\}$. The decomposition holds because $\sum_{0\le i\le\infty} \mathbb{1}\{\omega : \tau(\omega) = i\} = 1$. For each c in $\mathbb{R}^+$, the set $\{X > c\}$ is $\mathcal{F}_\tau$-measurable and, for $i \in \mathbb{N}_0$,

$$\{X_i > c\} = \{X > c\}\{t = i\} = \{X > c\}\{t \le i\} - \{X > c\}\{t \le i - 1\},$$

which belongs to $\mathcal{F}_i$. This shows that X_i is $\mathcal{F}_\tau$-measurable for each i in $\mathbb{N}_0$. For $i = \infty$ the result is even easier because $\mathcal{F}_\tau \subseteq \mathcal{F}_\infty$.

Conversely, if X has the stated decomposition then it is enough to show that each $Y_i := X_i\mathbb{1}\{\tau = i\}$ is $\mathcal{F}_\tau$-measurable. Clearly Y_i is $\mathcal{F}_i$-measurable. For each $c > 0$ and each $t \in \mathbb{N}_0$,

$$\{Y_i > c\}\{\tau \le t\} = \{X_i > c\}\{\tau = i\}\{\tau \le t\}.$$

This set is empty if $i > t$. If $i \le t$ it equals $\{X_i > c\}\{\tau = i\}$, which belongs to $\mathcal{F}_i \subseteq \mathcal{F}_t$. Thus $\{Y_i > c\}\{\tau \le t\} \in \mathcal{F}_t$ for each $t \in \mathbb{N}_0$, which is the test for showing that the $\mathcal{F}_\infty$-measurable set $\{Y_i > c\}$ belongs to $\mathcal{F}_\tau$ for each c in $\mathbb{R}^+$.

*[2] *Suppose $\{(X_i, \mathcal{F}_i) : i = 0, 1, \dots, n\}$ is a martingale with $\mathbb{P}X_i^2 < \infty$ for each i. Define $M := \max_{i\le n}|X_i|$.*

(i) *For each $t > 0$ define $\sigma(t) := \sigma_t(\omega) = n \wedge \inf\{i : |X_i(\omega)| \ge t\}$. Show that $\sigma(t)$ is a stopping time for which*

$$\{M \ge t\} = \{|X_{\sigma(t)}| \ge t\} \le |X_{\sigma(t)}|\{|X_{\sigma(t)}| \ge t\}/t.$$

Solution: *Note that $\{\sigma_t \le n\} = \Omega \in \mathcal{F}_n$ and*

$$\{\sigma_t \le k\} = \cup_{i\le k}\{|X_i| \ge t\} \in \mathcal{F}_k \qquad \text{for } k < n.$$

For the pointwise result, note that $M(\omega) \ge t$ iff there exists an $i \le n$ for which $|X_i(\omega)| \ge t$. The stopping time $\sigma_t(\omega)$ picks out the first such i. Conversely, because $\sigma_t(\omega)$ takes values in $\{0, 1, \dots, n\}$, if $|X_{\sigma_t(\omega)}(\omega)| \ge t$ then $M(\omega) \ge |X_i(\omega)| \ge t$ for $i = \sigma_t(\omega)$. The tricky *bit is handling the case where $|X_i(\omega)| < t$ for all i. In that case, $\sigma_t(\omega) = n$ but, fortunately $|X_n(\omega)| < t$.*

The inequality comes from the fact, for any random variable Y, that $|Y(\omega)|/t \geq 1$ if $|Y(\omega)| \geq t$.

(ii) *Use the stronger form of the Stopping Time Lemma to deduce that*

$$t\mathbb{P}\{M \geq t\} \leq \mathbb{P}|X_n|\{M \geq t\} \qquad \text{for each } t > 0.$$

SOLUTION: *The set $F = \{M \geq t\} = \{|X_{\sigma_t}| \geq t\}$ belongs to $\mathcal{F}_{\sigma_t}$ because X_{σ_t} is $\mathcal{F}_{\sigma_t}$-measurable. Note that $\{(|X_i|, \mathcal{F}_i) : i = 0, 1, \ldots, n\}$ is a submartingale and $0 \leq \sigma_t \leq \tau \equiv n$ always. By the STL,*

$$\mathbb{P}|X_{\sigma_t}|F \leq \mathbb{P}|X_n|F.$$

Note that part (i) gives $\{M \geq t\} = F \leq |X_{\sigma_t}|F/t$.

(iii) *Integrate the last inequality with respect to t then invoke the Hölder inequality to conclude that $\mathbb{P}M^2 \leq 4\mathbb{P}X_n^2$. Hint: If you plan on dividing by $\sqrt{\mathbb{P}M^2}$ you should explain why this quantity is neither zero nor infinite.*

SOLUTION:

$$\begin{aligned}
\mathbb{P}M^2 &= \mathbb{P}^\omega \int_0^\infty 2t\{\omega : M(\omega) \geq t\}\, dt \\
&= \int_0^\infty 2t\mathbb{P}^\omega\{\omega : M(\omega) \geq t\}\, dt \qquad \text{by Tonelli} \\
&\leq \int_0^\infty 2\mathbb{P}|X_n|\{\omega : M(\omega) \geq t\}\, dt \qquad \text{by part (ii)} \\
&= 2\mathbb{P}\left(|X_n| \int_0^\infty \{\omega : M(\omega) \geq t\}\, dt\right) \qquad \text{by Tonelli} \\
&= 2\mathbb{P}|X_n|M \\
&\leq 2\sqrt{(\mathbb{P}X_n^2)(\mathbb{P}M^2)} \qquad \text{by Hölder (or Cauchy-Schwarz).}
\end{aligned}$$

Note that the asserted inequality is trivial if $\mathbb{P}M^2 = 0$. Also

$$\mathbb{P}M^2 \leq \sum\nolimits_{i=0}^n \mathbb{P}X_i^2 < \infty.$$

Accordingly, there is no problem in dividing both sides by $\sqrt{\mathbb{P}M^2}$ then squaring both sides of the resulting inequality.

*[3] *Suppose $\{(X_t, \mathcal{F}_t) : t \in \mathbb{N}_0\}$ is a martingale with $\Gamma^2 := \sup_{t\in\mathbb{N}_0} \mathbb{P}X_t^2 < \infty$. Define $\mathcal{F}_\infty = \sigma\left(\cup_{t\in\mathbb{N}_0}\mathcal{F}_t\right)$.*

(i) *Show that $\mathbb{P}(X_\ell - X_k)X_k = 0$ if $k < \ell$. Deduce that $\mathbb{P}X_t^2$ increases to Γ^2 as $t \to \infty$.*

SOLUTION: *As $X_\ell \in \mathcal{L}^2(\mathcal{F})$ we have $\mathbb{P}_{\mathcal{F}_k} X_\ell \in \mathcal{L}^2(\mathcal{F}_k)$ (it is the orthogonal projection of X_ℓ onto $\mathcal{L}^2(\mathcal{F}_k)$) and $X_\ell - \mathbb{P}_{\mathcal{F}_k} X_\ell$ is orthogonal to $\mathcal{L}^2(\mathcal{F}_k)$. The martingale property gives $X_k = \mathbb{P}_{\mathcal{F}_k} X_\ell$ a.e. $[\mathbb{P}]$. Expand a square then take expected values to get*

$$\mathbb{P}X_\ell^2 = \mathbb{P}X_k^2 + 2\langle X_\ell - X_k, X_k \rangle + \mathbb{P}(X_\ell - X_k)^2.$$

The orthogonality kills the cross-product term. The resulting inequality shows that $\mathbb{P}X_\ell^2 \geq \mathbb{P}X_k^2$ if $\ell > k$. The sequence $\{\mathbb{P}X_k^2 : k \in \mathbb{N}_0\}$ increases to its supremum.

(ii) For ℓ and k in $\mathbb{N}_0$ with $\ell > k$ and $\delta > 0$ show that

$$\begin{aligned} \Gamma^2 \geq \mathbb{P}X_\ell^2 &= \mathbb{P}X_k^2 + \mathbb{P}(X_\ell - X_k)^2 \\ &\geq \Gamma^2 - \delta + \mathbb{P}(X_\ell - X_k)^2 \qquad \text{if } k \text{ is large enough.} \end{aligned}$$

Deduce that $\{X_t : t \in \mathbb{N}_0\}$ is a Cauchy sequence in $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$, which converges in $\mathcal{L}^2$ to some $X_\infty \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$.

SOLUTION: *We already have the first line. The second line comes from the convergence of $\mathbb{P}X_k^2$ to the finite Γ^2.*

(iii) Show that $\|X_\infty\|_2 = \Gamma$.

SOLUTION: *Use $|\, \|X_\infty\|_2 - \|X_k\|_2 \,| \leq \|X_k - X_\infty\|_2$.*

(iv) For positive integers $k < \ell$, use Problem [2] to show that

$$\mathbb{P}\sup\nolimits_{k \leq t \leq \ell} |X_t - X_k|^2 \leq 4\mathbb{P}|X_\ell - X_k|^2.$$

Let ℓ tend to ∞ to deduce that, for each fixed k,

$$\mathbb{P}\sup\nolimits_{t \geq k} |X_t - X_k|^2 \leq 4\epsilon_k^2 := 4\mathbb{P}|X_\infty - X_k|^2.$$

SOLUTION: *Apply Doob's inequality (from Problem 2) to the martingale*

$$\{(X_t - X_k, \mathcal{F}_t) : t = k, k+1, \dots, \ell\}$$

to get the first inequality.

As $\ell \to \infty$ the random variable $\sup_{k \leq t \leq \ell} |X_t - X_k|^2$ increases to $\sup_{t \geq k} |X_t - X_k|^2$. Appeal to Monotone Convergence. For the right-hand side use

$$|\, \|X_\ell - X_k\|_2 - \|X_\infty - X_k\|_2 \,| \leq \|X_\ell - X_\infty\|_2 \to 0 \qquad \text{as } \ell \to \infty.$$

(v) Define $Y_k := \sup_{t \geq k} |X_t - X_\infty|$. Show that

$$\mathbb{P}Y_k \leq \left\| \sup\nolimits_{t \geq k} |X_t - X_k| \right\|_2 + \|X_k - X_\infty\|_2 \leq 3\epsilon_k.$$

SOLUTION: *Start from*

$$|X_t - X_\infty| \le |X_t - X_k| + |X_k - X_\infty|.$$

Take $\sup_{t\ge k}$ *on both sides to get*

$$Y_k \le \sup_{t\ge k} |X_t - X_k| + |X_k - X_\infty|.$$

Then argue that

$$\mathbb{P}Y_k = \|Y_k\|_1 \le \|Y_k\|_2 \le \left\|\sup_{t\ge k} |X_t - X_k|\,\right\|_2 + \|X_k - X_\infty\|_2 .$$

Use (iv) to bound the right-hand side.

(vi) Prove that $Y_k \downarrow 0$ a.e. $[\mathbb{P}]$. *Deduce that* $X_t \to X_\infty$ a.e. $[\mathbb{P}]$ *as* $t \to \infty$.

SOLUTION: *As k increases the set of t's over which the sup is taken decreases. The non-negative random variables must decrease pointwise to a limit* $Y_\infty \ge 0$. *From (v),* $\mathbb{P}Y_\infty \le 3\epsilon_k$ *for every k, which implies* $Y_\infty = 0$ a.e. $[\mathbb{P}]$.

For the second bit use $Y_k \ge |X_k - X_\infty|$.

**(vii) Show that* $X_t = \mathbb{P}(X_\infty \mid \mathcal{F}_t)$ a.e. $[\mathbb{P}]$.

SOLUTION: *For* $F \in \mathcal{F}_t$ *and* $\ell > t$,

$$|\mathbb{P}(X_t F) - \mathbb{P}(X_\infty F)| = |\mathbb{P}(X_\ell F) - \mathbb{P}(X_\infty F)| \le \|X_\ell - X_\infty\|_2 .$$

Let ℓ tend to infinity.

[4] *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $\mathcal{G}$ *be a sub-sigma-field of* $\mathcal{F}$. *Suppose* $\{X_n : n \in \mathbb{N}\}$ *is a sequence of random variables that converges to* 0 a.e. $[\mathbb{P}]$. *Suppose also that* $|X_n| \le W \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ *for all n.*

Define $Z_i = \mathbb{P}_{\mathcal{G}} X_i$ *and* $S_n = \sup_{i\ge n} |X_i|$ *and* $Y_n = \sup_{i \ge n} |Z_i|$. *Prove that*

$$\mathbb{P}Y_n \le \mathbb{P}S_n \to 0 \qquad \text{as } n \to \infty.$$

Deduce that $\mathbb{P}_{\mathcal{G}} X_n \to 0$ a.e. $[\mathbb{P}]$.

[5] *(HARD) Suppose* $\{Z_i : i \in \mathbb{N}_0\}$ *is a sequence of random variables defined on* Ω *and* $\mathcal{F}_n = \sigma(Z_0, Z_1, \dots, Z_n)$ *for* $n \in \mathbb{N}_0$. *Let τ be a stopping time for that filtration.*

(i) Explain why every $\mathcal{F}_n$-measurable (real valued) random variable Y can be written in the form $Y(\omega) = g_n(Z_0(\omega), \dots, Z_n(\omega))$ *for some* $\mathcal{B}(\mathbb{R}^{n+1})$*-measurable function* g_n. *Hint: Lecture 3.*

(ii) Define $X_i = Z_{\tau\wedge i}$ *and* $\mathcal{G} := \sigma(X_i : i \in \mathbb{N}_0)$. *Prove that X_i is $\mathcal{F}_\tau$-measurable. Deduce that* $\mathcal{G} \subseteq \mathcal{F}_\tau$. *Hint: Split* $\{X_i \in B\}\{\tau \le n\}$ *into contributions from various sets* $\{\tau = j\}$.

(iii) Prove that τ is $\mathcal{G}$-measurable. Hint: $\{\tau = 0\} = g_0(Z_0) = g_0(X_0)$ *and* $\{\tau = 1\} = g_1(Z_0, Z_1) = g_1(Z_0, Z_1)\{\tau \ge 1\} = g_1(X_0, X_1)\{\tau = 0\}^c$.

(iv) Show that $\mathcal{F}_\tau \subseteq \mathcal{G}$. *Hint: If* $F \in \mathcal{F}_\tau$ *consider sets* $F\{\tau = j\}$ *for* $j \in \mathbb{N}_0$.