Statistics 330b/600b, Math 330b spring 2018
Homework # 11
Due: Thursday 19 April

*[4] I have a collection of m different coins, the ith of which lands heads with probability θ_i when tossed. Let $\mathbb{Q}$ be a probability measure on $\{1, 2, \dots, m\}$ for which $\mathbb{Q}\{i\} = p_i$ for $i = 1, \dots, m$, where $p_i > 0$ for each i. I generate a sequence of random variables $X_1, X_2, \dots$ as follows. First generate an observation T from $\mathbb{Q}$. If $T = i$ then toss the ith coin repeatedly, recording $X_j = 1$ if the jth toss lands heads and $X_j = 0$ for tails.

(i) Show that $\mathbb{P}X_1 = \overline{\theta} := \sum_{i=1}^m p_i\theta_i$. Hint: Condition.

(ii) For each set $\{n_1, \dots, n_k\}$ of k distinct positive integers find $\mathbb{P}(\prod_{j=1}^k X_{n_j})$.

*(iii) Show that the sequence $(X_n, n \in \mathbb{N})$ is exchangeable (UGMTP Definition 6.49).

(iv) Show that $\text{cov}(X_1, X_2) = 0$ if and only if $\theta_i = \overline{\theta}$ for every i.

*(v) Explain why $X_1, X_2, \dots$ are independent if and only if $\theta_i = \overline{\theta}$ for every i.

Remark. The ill-fated problem [4] has been causing more problems. You could interpret it the following way.

Suppose $\mathbb{P}_1, \dots, \mathbb{P}_m$ are probability measures on $(\Omega, \mathcal{F})$ and $X_1, X_2, \dots$ are measurable maps from Ω into $\{0, 1\}$. Under $\mathbb{P}_i$, the X_j's are independent, each with distribution $\text{Ber}(\theta_i)$. Define $\mathbb{P} = \sum_{i=1}^m p_i\mathbb{P}_i$.

Part (iii) effectively asks you to prove that, for each k and each permutation σ of $(1, 2, \dots, k)$ and all choices of $\alpha_1, \dots, \alpha_k$ from $\{0, 1\}$,

$$\mathbb{P}\{X_1 = \alpha_1, X_2 = \alpha_2, \dots, X_k\} = \mathbb{P}\{X_{\sigma(1)} = \alpha_1, X_{\sigma(2)} = \alpha_2, \dots, X_{\sigma(k)}\}.$$

Hint: Note that $\mathbb{1}\{X_j = 1\} = X_j$ and $\mathbb{1}\{X_j = 0\} = 1 - X_j$. That leads to simplifications such as

$$\begin{aligned}\mathbb{1}\{X_1 = 1, X_2 = 0, X_3 = 1, X_4 = 0\} &= X_1(1 - X_2)X_3(1 - X_4) \\ &= X_1X_3 - \dots + X_1X_2X_3X_4.\end{aligned}$$

SOLUTION: In the original statement of the problem $\mathbb{Q}$ was supposed to be the distribution of T and $\mathbb{P}_i$ was supposed to be thought of as $\mathbb{P}(\cdot \mid T = i)$.

The suggested method of solution for (iii) was unnecessarily complicated. Instead argue, for each k and each choice of $\alpha_1, \dots, \alpha_k$ from $\{0, 1\}$,

$$\begin{aligned}\mathbb{P}\{X_1 = \alpha_1, \dots, X_k = \alpha_k\} &= \sum\nolimits_i p_i\mathbb{P}_i\{X_1 = \alpha_1, \dots, X_k = \alpha_k\} \\ &= \sum\nolimits_i p_i\theta_i^h(1 - \theta_i)^{1-h} \qquad \text{where } h = \sum\nolimits_{i=1}^k \alpha_i.\end{aligned}$$

If $t = \sigma^{-1}$, that is, if $\sigma(i) = j$ iff $\tau(j) = i$, then

$$\mathbb{P}\{X_{\sigma(1)} = \alpha_1, \dots, X_{\sigma(k)} = \alpha_k\} = \mathbb{P}\{X_1 = \alpha_{\tau(1)}, \dots, X_k = \alpha_{\tau(k)}\},$$

which again equals $\sum_i p_i\theta_i^h(1 - \theta_i)^{1-h}$ because h is unaffected by the τ permutation. That gives the exchangeability.

Specialize to $k = 1$ and $\alpha_1 = 1$ to get $\mathbb{P}X_1 = \mathbb{P}\{X_1 = 1\} = \overline{\theta}$.

Specialize to $\alpha_1 = \alpha_2 = \dots = \alpha_k = 1$ to get $\mathbb{P}(X_1 \dots X_k) = \sum_i p_i\theta_i^k$. Exchangeability gives the same value for distinct $n_1, \dots, n_k$.

Consequently,

$$\text{cov}(X_1, X_2) = \mathbb{P}(X_1X_2) - (\mathbb{P}X_1)(\mathbb{P}X_2) = \left(\sum\nolimits_i p_i\theta_i^2\right) - \overline{\theta}^2 = \sum\nolimits_i p_i(\theta_i - \overline{\theta})^2.$$

Clearly $\text{cov}(X_1, X_2) = 0$ iff $\theta_i = \overline{\theta}$ for every i. A non-zero covariance implies dependence. And if $\theta_i = \overline{\theta}$ for each i then $X_1, X_2, \dots$ are independent $\text{Ber}(\overline{\theta})$ distributed under $\mathbb{P}$.