## 602 PROJECT 1

[1] Suppose $P$ is a probability measure on $\mathcal{B}(\mathbb{R})$ with mean 0 and variance 1 . You have seen that, for any given Brownian motion $\{B(t): t \geq 0\}$ (continuous paths, started at zero), there exists a (randomized) stoppping time $\tau$ for which $B(\tau)$ has distribution $P$ with $\mathbb{P} \tau=1$. Moreover, if we repeat the construction we can get a whole sequence of independent, identically distributed times $\tau_{1}, \tau_{2}, \ldots$ giving stopping times $T_{i}=\tau_{1}+\ldots+\tau_{i}$ for $B$ such that the random variables $Y_{i}=B\left(T_{i}\right)-B\left(T_{i-1}\right)$ are independent, each with distribution $P$. (Define $T_{0}:=0$.)
(i) By the SLLN, $\xi_{i}:=\left|T_{i} / i-1\right| \rightarrow 0$ almost surely. Use this result to show that

$$
\mathbb{P}\left\{\max _{i \leq 2 n}\left|T_{i}-i\right| / n>\delta\right\} \rightarrow 0 \quad \text { as } n \rightarrow \infty, \text { for each } \delta>0
$$

(ii) Use the uniform continuity on bounded intervals of the sample paths of $B$ to show that

$$
R(\delta):=\sup \{|B(s)-B(t)|: 0 \leq t \leq 1, \quad|s-t|<\delta\} \rightarrow 0 \quad \text { in probability as } \delta \rightarrow 0
$$

[2] Argue similarly, for each fixed $n$ with the Brownian motion $B_{n}(t)=\sqrt{n} B(t / n)$, to find stopping times $0 \leq T_{1, n} \leq T_{2, n} \leq \ldots$ with each of the independent increments $Y_{i, n}:=B_{n}\left(T_{i, n}\right)-B_{n}\left(T_{i-1, n}\right)$ having distribution $P$. Note that each of the independent increments $\tau_{i, n}:=T_{i, n}-T_{i-1, n}$ has the same distribution as the $\tau_{1}$ from Problem [1]. Define $S_{i, n}=B_{n}\left(T_{i, n}\right)$
(a) Define $\left\{X_{n}(t): t \geq 0\right\}$ to be the partial-sum process, which interpolates linearly between the points $\left(i / n, S_{i, n} / \sqrt{n}\right)$ for $i \in \mathbb{N}_{0}$.
(b) Define $\lambda_{n}$ to be the continuous, increasing function that interpolates linearly between the points $\left(i / n, T_{i, n} / n\right)$ for $i \in \mathbb{N}_{0}$.
Now use the results from Problem [1] to show that

$$
\begin{aligned}
& \sup _{0 \leq t \leq 1}\left|\lambda_{n}(t)-t\right| \rightarrow 0 \quad \text { in probability as } n \rightarrow \infty \\
& \sup _{0 \leq t \leq 1}\left|B\left(\lambda_{n}(t)\right)-B(t)\right| \rightarrow 0 \quad \text { in probability as } n \rightarrow \infty \\
& \sup _{0 \leq t \leq 1}\left|B\left(\lambda_{n}(t)\right)-X_{n}(t)\right| \rightarrow 0 \quad \text { in probability as } n \rightarrow \infty
\end{aligned}
$$

Conclude that $\sup _{0 \leq t \leq 1}\left|X_{n}(t)-B(t)\right| \rightarrow 0$ in probability as $n \rightarrow \infty$.
[3] Give a complete proof for the coupling of the uniform empirical process with a Brownian Bridge. You may assume the result about the representation of uniform order statistics as ratios of partial sums of independent standard exponentials. Try to tidy up my argument for obtaining upper and lower bounds for $G_{n}^{-1}(t)$.

