Appendix G

Minimax theorem

[§general] ## 1. A general minimax theorem

mmax.thm   <1>   **Theorem.** *Let K be a compact convex subset of a Hausdorff topological vector space $\mathcal{X}$, and C be a convex subset of a vector space $\mathcal{Y}$. Let f be a*

extend to $\mathbb{R} \cup \{\infty\}$ valued f?   *real-valued function defined on $K \times C$ such that*

     (i) $x \mapsto f(x, y)$ is convex and lower-semicontinuous for each y,

     (ii) $y \mapsto f(x, y)$ is concave for each x.

Then

$$\inf_{x \in K} \sup_{y \in C} f(x, y) = \sup_{y \in C} \inf_{x \in K} f(x, y).$$

Proof. Note that assumption (i) means precisely that $K_{y,t} := \{x \in K : f(x, y) \leq t\}$ is compact and convex, for each fixed $t \in \mathbb{R}$ and $y \in C$.

Clearly the left-hand side of the asserted equality is $\geq$ the right-hand side. It therefore suffices to prove that if M is a real number for which $M \geq$ right-hand side then, for each $\epsilon > 0$, the left-hand side is $\leq M + \epsilon$. The assumption on M means that $\inf_{x \in K} f(x, y) \leq M$, and hence $K_{y,M+\epsilon} \neq \emptyset$, for every y. If we can prove that $\cap_{y \in \mathcal{Y}} K_{y,M+\epsilon} \neq \emptyset$ then there will exist an x_0 for which $f(x_0, y) \leq M + \epsilon$ for every y, implying that $\sup_y f(x_0, y) \leq M + \epsilon$.

Replacing f by $f - (M + \epsilon)$, we reduce to the case where $M + \epsilon = 0$ and $\inf_{x \in K} f(x, y) \leq -\epsilon$ and $K_{y,0} \neq \emptyset$, for each y. We need to show that $\cap_{y \in \mathcal{Y}} K_{y,0} \neq \emptyset$. Compactness of each $K_{y,0}$ simplifies the task to showing that $\cap_{y \in \mathcal{Y}_0} K_{y,0} \neq \emptyset$ for each finite subset $\mathcal{Y}_0$ of $\mathcal{Y}$. An inductive argument will then reduce even further to the case where $\mathcal{Y}_0 := \{y_1, y_2\}$, a two-point set, which is the case that I consider first.

Abbreviate $K_{y_i,0}$ to K_i, and $f(x, y_i)$ to $f_i(x)$. Thus $K_i = \{x : f_i(x) \leq 0\}$. For the purposes of obtaining a contradiction, suppose $K_1 \cap K_2 = \emptyset$. The contradiction will appear if we find a number α in $[0, 1]$ for which

alpha.def   <2>
$$(1 - \alpha) f_1(x) + \alpha f_2(x) \geq 0 \qquad \text{for all } x \text{ in } K,$$

for then the concavity of $f(x, \cdot)$ would give the lower bound $\inf_{x \in K} f(x, y_\alpha) \geq 0$, where $y_\alpha := (1 - \alpha) y_1 + \alpha y_2$.

Inequality <2> is trivial if $x \notin K_1 \cup K_2$, for then both $f_1(x) > 0$ and $f_2(x) > 0$. For it to hold at each x in K_1 we would need

lower.alpha   <3>
$$\alpha \geq \sup_{x_1 \in K_1} \frac{-f_1(x_1)}{f_2(x_1) - f_1(x_1)}.$$

Notice that the supremum on the right-hand side is ≥ 0. For inequality <2> to hold at each x in K_2 we would need

upper.alpha   <4>
$$\alpha \leq \inf_{x_2 \in K_2} \frac{f_1(x_2)}{f_1(x_2) - f_2(x_2)}.$$
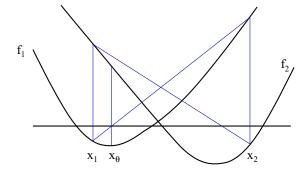
Notice that the infimum on the right-hand side is ≤ 1. There exists an α satisfying both constraints <3> and <4> if and only if

$$\frac{-f_1(x_1)}{f_2(x_1) - f_1(x_1)} \leq \frac{f_1(x_2)}{f_1(x_2) - f_2(x_2)} \qquad \text{for all } x_1 \in K_1 \text{ and } x_2 \in K_2.$$

That is, α exists if and only if

f1f2   <5> $(-f_1(x_1))\,(-f_2(x_2)) \leq f_1(x_2)f_2(x_1) \qquad \text{for all } x_1 \in K_1 \text{ and } x_2 \in K_2.$

This inequality involves the values of the convex functions only along the line joining x_1 and x_2; it is essentially a one-dimensional result.

The inequality <5> is trivial when either $f_1(x_1) = 0$ or $f_2(x_2) = 0$. We need only consider a pair with $f_1(x_1) < 0$ and $f_2(x_2) < 0$. Define θ in $(0, 1)$ as the value for which $(1 - \theta)f_1(x_1) + \theta f_1(x_2) = 0$, then define $x_\theta := (1 - \theta)x_1 + \theta x_2$. By convexity, $f_1(x_\theta) \leq 0$, implying that $x_\theta \in K_1$ and $(1 - \theta)f_2(x_1) + \theta f_2(x_2) \geq f_2(x_\theta) > 0$. Thus

$$\frac{-f_1(x_1)}{f_1(x_2)} = \frac{\theta}{1 - \theta} < \frac{f_2(x_1)}{-f_2(x_2)},$$

which gives <5>.

Existence of an α satisfying constraints <3> and <4> now follows, which, via the contradiction, lets us conclude that $K_1 \cap K_2 \neq \emptyset$.

To extend the conclusion to an intersection of finitely many sets $K_{y_i,0}$, for $i = 1, 2, \ldots, m$, first invoke the result for pairs to see that $K'_i := K_{y_1,0} \cap K_{y_i,0} \neq \emptyset$ for $i = 2, \ldots, m$, then repeat the pairwise argument with f restricted to $K_{y_1,0} \times C$. And so on. After $m - 1$ repetitions we reach the desired conclusion,
☐   that $\cap_{i \leq m} K_{y_i,0} \neq \emptyset$.

convex.hull   <6>   **Example.**   Let $\mathcal{P}$ and $\mathcal{Q}$ be collections of probability measures on the same space $(\mathcal{X}, \mathcal{A})$. Write co$(\mathcal{P})$ and co$(\mathcal{Q})$ for their convex hulls. That is, co$(\mathcal{P})$ consists of all finite linear combinations $\sum_i \alpha_i \mathbb{P}_i$, with $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$ and $\mathbb{P}_i \in \mathcal{P}$, with a similar definition for co$(\mathcal{Q})$. Write $\mathbb{T}$ for the collection of all tests, that is, functions ψ in $\mathbb{M}(\mathcal{X}, \mathcal{A})$ for which $0 \leq \psi \leq 1$. If $\psi \in \mathbb{T}$, the function $\bar{\psi} := 1 - \psi$ is also a test.

If both $\mathcal{P}$ and $\mathcal{Q}$ are dominated by some sigma-finite measure λ, the Minimax Theorem will show that

lecam.mmax   <7> $\displaystyle \inf_{\psi \in \mathbb{T}} \sup\{\mathbb{P}\psi + \mathbb{Q}\bar{\psi} : \mathbb{P} \in \mathcal{P}, \mathbb{Q} \in \mathcal{Q}\} = \sup\{\|\widetilde{\mathbb{P}} \wedge \widetilde{\mathbb{Q}}\| : \widetilde{\mathbb{P}} \in \text{co}(\mathcal{P}), \widetilde{\mathbb{Q}} \in \text{co}(\mathcal{Q})\}.$

Before proving the equality, first note that the left-hand side does not change if we replace both $\mathcal{P}$ and $\mathcal{Q}$ by their convex hulls, because

$$\sum_i \alpha_i \mathbb{P}_i \psi + \sum_j \beta_j \mathbb{Q}_j \bar{\psi} = \sum_{i,j} \alpha_i \beta_j \left(\mathbb{P}_i \psi + \mathbb{Q}_j \bar{\psi}\right).$$

Thus there is no loss of generality in assuming both $\mathcal{P}$ and $\mathcal{Q}$ are convex. The set C of all signed measures of the form $\nu = \mathbb{P} - \mathbb{Q}$, with $\mathbb{P} \in \mathcal{P}$ and $\mathbb{Q} \in \mathcal{Q}$, is

then convex. Next note that $\|\widetilde{\mathbb{P}} \wedge \widetilde{\mathbb{Q}}\| = \inf_{\psi \in \mathbb{T}} \left(\mathbb{P}\psi + \mathbb{Q}\bar{\psi} \right)$. After subtraction of 1 from both sides, the assertion <7> can be written as

$$\inf_{\psi \in \mathbb{T}} \sup_{\nu \in C} \nu\psi = \sup_{\nu \in C} \inf_{\psi \in \mathbb{T}} \nu\psi.$$

Notice that $\nu\psi$ is linear, separately, in both ν and ψ.

We can identify each ν with an element of $L^1(\lambda)$, the set of equivalence classes of λ-integrable functions. Two tests that differ only on a λ-neglible set give the same value to $\nu\psi$, for every ν in C. Thus we can identify $\mathbb{T}$ with a closed subset of the linear space $L^\infty(\lambda)$ of λ-equivalence classes of bounded functions. Under the weakest toplogy on $L^\infty(\lambda)$ that makes $f \mapsto \lambda(fg)$ continuous, for each g in $L^1(\lambda)$, the set $\mathbb{T}$ is compact and convex, and the map $\psi \mapsto \nu\psi$ is continuous for each ν in C.

cite Dunford & Schwartz (1958)?■

Assertion <7> is a special case of the Minimax Theorem <1>.

□

[§]   **2.   Notes**

Sources: Kneser (1952)? Fan (1953)? Sion (1958)? See Millar (1983, page 92). Possibly also some handwritten notes from other lectures by Millar.

Le Cam (1973) stated the result from Example <6> with a reference to Kraft (1955) for the proof. Kraft attributed the result and the proof to Le Cam. When the families are not dominated, the proof breaks down. Le Cam (1986, page 476) stated generalized forms of the result, based on compactifications of either $\mathbb{T}$ or C.

REFERENCES

Dunford, N. & Schwartz, J. T. (1958), *Linear Operators, Part I: General Theory*, Wiley.

Fan, K. (1953), 'Minimax theorems', *Proceedings of the National Academy of Sciences* **39**, 42–47.

Kneser, H. (1952), 'Sur un théorème fondamental de la théorie des jeux', *Comptes Rendus de l'Academie des Sciences, Paris* **234**, 2418–2420.

Kraft, C. (1955), 'Some conditions for consistency and uniform consistency of statistical procedures', *University of California Publications in Statistics* **2**, 125–142.

Le Cam, L. (1973), 'Convergence of estimates under dimensionality restrictions', *Annals of Statistics* **1**, 38–53.

Le Cam, L. (1986), *Asymptotic Methods in Statistical Decision Theory*, Springer-Verlag, New York.

Millar, P. W. (1983), 'The minimax principle in asymptotic statistical theory', *Springer Lecture Notes in Mathematics* **976**, 75–265.

Sion, M. (1958), 'On general minimax theorems', *Pacific Journal of Mathematics* **8**, 171–176.