Brownian motion: introduction

A standard Brownian motion on $\mathbb{R}^+$ for a filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$ is an adapted process for which

(i) all sample paths are continuous

(ii) $X(0, \omega) = 0$ for all ω

(iii) for each pair s, t with $0 \le s \le t$,

$$X_t - X_s \qquad \text{is } N(0, t-s) \text{ distributed independent of } \mathcal{F}_s$$

Properties (ii) and (iii) together imply: for each $0 \le t_1 \le t_2 \le \ldots \le t_k$, the random vector $(X_{t_1}, \ldots, X_{t_k})$ has a multivariate normal distribution with zero means and covariances given by

$$\text{cov}(X_s, X_t) = \min(s, t)$$

Abbreviate $\mathbb{P}(\ldots \mid \mathcal{F}_t)$ to $\mathbb{P}_t(\ldots)$.

Useful facts: some rigorous proofs to follow

(i) For a fixed $\tau \ge 0$ define $Z_t = B_{\tau+t} - B_\tau$ for $t \ge 0$. Then Z is a standard Brownian motion independent of $\mathcal{F}_\tau$.

(ii) (Strong Markov property) Same assertion as in (i) except that τ is a stopping time.

(iii) (Time reversal) Define $Z_t = tB_{1/t}$ for $t > 0$, with $Z_0 = 0$. Then $\{Z_t : t \in \mathbb{R}^+\}$ is a also a standard Brownian motion.

(iv) Both $\{(B_t, \mathcal{F}_t) : t \in \mathbb{R}^+\}$ and $\{(B_t^2 - t, \mathcal{F}_t) : t \in \mathbb{R}^+\}$ are martingales.

(v) For each real θ, the process $Y_t = \exp\left(\theta X_t - \frac{1}{2}\theta^2 t\right)$ is a martingale. (For complex θ, would it be a complex martingale?)

<1> **Lévy's martingale characterization of Brownian motion.** *Suppose $\{X_t : 0 \le t \le 1\}$ is a martingale with continuous sample paths and $X_0 = 0$. Suppose also that $X_t^2 - t$ is a martingale. Then X is a Brownian motion.*

Heuristics of the proof that $X_1 \sim N(0, 1)$. The two martingale assuptions give two properties of the increment $\Delta X = X_t - X_s$, for $s < t$:

<2> $$\mathbb{P}_s \Delta X = 0 \qquad \text{and} \qquad \mathbb{P}_s(\Delta X)^2 = t - s.$$

Let $f(x, t)$ be a smooth function of two arguments, $x \in \mathbb{R}$ and $t \in [0, 1]$. Define

$$f_x = \frac{\partial f}{\partial x} \qquad \text{and} \qquad f_{xx} = \frac{\partial^2 f}{\partial^2 x} \qquad \text{and} \qquad f_t = \frac{\partial f}{\partial t}.$$

Let $h = 1/n$ for some large positive integer n. Define $t_i = ih$ for $i = 0, 1, \ldots, n$. Write $\Delta_i X$ for $X(t_i + h) - X(t_i)$. Then

$$\begin{aligned}
&\mathbb{P}f(X_1, 1) - \mathbb{P}f(X_0, 0) \\
&\qquad = \sum_{i<n} \left(\mathbb{P}f(X_{t_i+h}, t_i + h) - \mathbb{E}f(X_{t_i}, t_i)\right) \\
&\qquad \approx \sum_{i<n} \mathbb{P}\left((\Delta_i X) f_x(X_{t_i}, t_i) + \tfrac{1}{2}(\Delta_i X)^2 f_{xx}(X_{t_i}, t_i) + h f_t(X_{t_i}, t_i)\right) \\
&\qquad = \sum_{i<n} \left(0 + \tfrac{1}{2}h\mathbb{P}f_{xx}(X_{t_i}, t_i) + h\mathbb{P}f_t(X_{t_i}, t_i)\right) \\
&\qquad \approx \int_0^1 \left(\tfrac{1}{2}\mathbb{P}f_{xx}(X_s, s) + \mathbb{P}f_t(X_s, s)\right) ds \qquad \text{if } h \text{ is small} \qquad \text{by <2>.}
\end{aligned}$$

We need to formalize the passage to the limit to get

$$\mathbb{P}f(X_1, 1) - \mathbb{P}f(X_0, 0) = \int_0^1 \left(\tfrac{1}{2}\mathbb{P}f_{xx}(X_s, s) + \mathbb{P}f_t(X_s, s)\right) ds.$$

Specialize to the case $f(x, s) = \exp\left(\theta x - \frac{1}{2}\theta^2 s\right)$, with θ a fixed constant. By direct calculation,

$$f_x = \theta f(x, s) \qquad \text{and} \qquad f_{xx} = \theta^2 f(x, s) \qquad \text{and} \qquad f_t = -\tfrac{1}{2}\theta^2 f(x, s)$$

Thus

$$\mathbb{P}e^{\theta X_1} e^{-\theta^2/2} - 1 = \int_0^1 0\, ds = 0.$$

That is, X_1 has the moment generating function $\exp(\theta^2/2)$, which identifies it as having a $N(0, 1)$ distribution. □

As you will see, we are effectively proving a martingale central limit theorem. Look at the handout ***martingaleCLT.pdf*** before we start on a rigorous proof.