DOLÉANS MEASURES

Notation

- $\mathcal{T}_1$ = the set of all [0, 1]-valued stopping times
- for $\sigma, \tau \in \mathcal{T}_1$,

$$((\sigma, \tau]] := \{(t, \omega) \in (0, 1] \times \Omega : \sigma(\omega) < t \leq \tau(\omega)\}$$
$$[[\sigma, \tau]] := \{(t, \omega) \in (0, 1] \times \Omega : \sigma(\omega) \leq t \leq \tau(\omega)\}$$

and so on.

1. Introduction

The construction (Project 4) of the isometric stochastic integral $H \bullet M$ with respect to a martingale $M \in \mathcal{M}^2[0, 1]$, at least for bounded, predictable H, depended on the existence of the Doléans measure μ on the predictable sigma-field $\mathcal{P}$ on $(0, 1] \times \Omega$. To make the map $H \mapsto H \bullet M_1$ an isometry between $\mathcal{H}_{\text{simple}}$ and a subset of $\mathcal{L}^2(\Omega, \mathcal{F}_1, \mathbb{P})$ we needed

<1> $$\mu(a, b] \times F = \mathbb{P}\{\omega \in F\}(M_b - M_a)^2 \qquad \text{for all } 0 \leq a < b \leq 1 \text{ and } F \in \mathcal{F}_a.$$

⇛ This property characterizes the measure μ because the collection of all predictable sets of the form $(a, b] \times F$ is $\cap f$-stable and it generates $\mathcal{P}$.

The sigma-field $\mathcal{P}$ is also generated by the set of all stochastic intervals $((0, \tau]]$ for $\tau \in \mathcal{T}_1$. The Doléans measure is also characterized by the property

⇛ $$\mu((0, \tau]] = \mathbb{P}(M_\tau - M_0)^2 \qquad \text{for all } \tau \in \mathcal{T}_1.$$

Notice that μ depends on M only through the submartingale $S_t := \left(M_t - M_0\right)^2$:

$$\mathbb{P}F(M_b - M_a)^2 = \mathbb{P}F(S_b - S_a) \qquad \text{for } F \in \mathcal{F}_a.$$

In fact, analogous measures can be defined for a large class of submartingales.

<2> **Definition.** *Let $\{S_t : 0 \leq t \leq 1\}$ be a cadlag submartingale. Say that a finite (countably-additive) measure μ_S, defined on the predictable sigma-field of $(0, 1] \times \Omega$, is the* ***Doléans measure*** *for S if $\mu((0, \tau]] = \mathbb{P}\left(S_\tau - S_0\right)$ for every τ in $\mathcal{T}_1$.*

REMARK.
If μ_S exists then $S_\tau - S_0$ must be integrable for every τ in $\mathcal{T}_1$. Note also that the definition is not affected if we replace S_t by $S_t - S_0$. Thus there is no loss of generality in assuming that $S_0 \equiv 0$.

I mentioned explicitly that μ_S must be countably-additive to draw attention to a subtle requirement on S for μ_S to exist, known somewhat cryptically as property [D]:

[D] $$\{S_\tau : \tau \in \mathcal{T}_1\} \text{ is uniformly integrable.}$$

<3> **Example.** If $M \in \mathcal{M}^2[0,1]$ then the submartingale $S_t = (M_t - M_0)^2$ has property $[D]$. Indeed, we know from Project 2 that $M_\tau = \mathbb{P}(M_1 \mid \mathcal{F}_\tau)$ for all $\tau \in \mathcal{T}_1$. Thus

$$0 \leq \left(M_\tau - M_0\right)^2 \leq \mathbb{P}\left(\left(M_1 - M_0\right)^2 \mid \mathcal{F}_\tau\right) \qquad \text{for each } \tau \text{ in } \mathcal{T}_1.$$

$\Rrightarrow$ Use the fact that $\{\mathbb{P}(\xi \mid \mathcal{G}) : \mathcal{G} \subseteq \mathcal{F}\}$ is uniformly integrable for each integrable random variable ξ to complete the argument. □

<4> **Example.** Let $\{B_t : 0 \leq t \leq 1\}$ be a standard Brownian motion. The submartingale $S_t := B_t^2$ has a very simple Doléans measure, characterized by

$$\mu_S(a,b] \otimes F = \mathbb{P}F\left(B_b^2 - B_a^2\right) = \mathbb{P}F(b-a) \qquad \text{for } F \in \mathcal{F}_a.$$

That is, $\mu_S = \mathfrak{m} \otimes \mathbb{P}$, with $\mathfrak{m}$ equal to Lebesgue measure on $\mathcal{B}[0,1]$. Of course μ_S has a further extension to the product sigma-field $\mathcal{B}[0,1] \otimes \mathcal{F}$.

A Poisson process $\{N_t : 0 \leq t \leq 1\}$ with intensity 1 shares with Brownian motion the independent increment property, but the increment $N_t - N_s$ has a Poisson$(t-s)$ distribution. The sample paths are constant, except for jumps of size 1 corresponding to points of the process. The process $\{N_t : 0 \leq t \leq 1\}$ is a $\Rrightarrow$ submartingale with respect to its natural filtration, with Doléans measure $\mathfrak{m} \otimes \mathbb{P}$, the same as the square of Brownian motion.

Clearly the Doléans measure does not uniquely determine the submartingale: both squared Brownian motion and the Poisson process have Doléans measure $\mathfrak{m} \otimes \mathbb{P}$. But the only square integrable martingale M with continuous sample paths and Doléans measure $\mu_M = \mathfrak{m} \otimes \mathbb{P}$ is Brownian motion: if $F \in \mathcal{F}_s$ and $s < t$ then $\mathbb{P}F(M_t^2 - M_s^2) = \mu_M(s,t] \otimes F = (t-s)\mathbb{P}F$, from which it follows that $M_t^2 - t$ is a martingale with respect to $\{\mathcal{F}_t\}$. It follows from Lévy's characterization that M is a Brownian motion. □

Problem [2] shows that if there exists a (countably-additive) Doléans measure μ_S then S has property [D]. The proof of the converse assertion is the main subject of this handout.

<5> **Theorem.** *Every cadlag submartingale $\{S_t : 0 \leq t \leq 1\}$ with property [D] has a countably additive Doléans measure μ_S.*

There are several ways to prove this assertion For example:

(i) Invoke an approximation by compact sets (Métivier 1982, Chapter 3).

(ii) For the square of a continuous $\mathcal{M}_0^2[0,1]$-martingale M, prove directly the existence of an increasing process A (the quadratic variation process) for which $M_t^2 - A_t$ is a martingale (Chung & Williams 1990, Section 4.4).

(iii) Do something very general, as in Dellacherie & Meyer (1982, §7.1).

I will present a different method, based on the identification of measures on $\mathcal{P}$ with a certain kind of linear functional defined on the vector space $\mathcal{H}_{\text{BddLip}}$ of all adapted, continuous processes H on $[0,1] \times \Omega$ for which there exists a finite constant C_H such that

(i) $|H(t,\omega)| \leq C_H$ for all (t,ω).

(ii) $|H(s,\omega) - H(t,\omega)| \le C_H|t-s|$ for all s, t, and ω.

⇛ It is easy to show that $\mathcal{H}_{\text{BddLip}}$ generates a sigma-field $\mathcal{P}_0$ on $[0,1]\times\Omega$ for which

$$\mathcal{P} = \{D \cap \big((0,1]\times\Omega\big) : D \in \mathcal{P}_0\}.$$

As a consequence of the Theorem stated in Section 3, an increasing linear functional $\mu : \mathcal{H}_{\text{BddLip}} \to \mathbb{R}$ is defined by the integral with respect to a finite, countably additive measure on $\mathcal{P}_0$ if and only if it is ***sigma-smooth at*** 0, that is,

$$\mu(h_n) \downarrow 0 \qquad \text{for each } \{h_n : n \in \mathbb{N}\} \subseteq \mathcal{H}_{\text{BddLip}} \text{ with } h_n \downarrow 0 \text{ pointwise.}$$

If $\mu\big(\{0\}\times\Omega\big) = 0$ then μ can also be thought of as a measure on $\mathcal{P}$.

2. The Doléans measure as a linear functional

Let $\{S_t : 0 \le t \le 1\}$ be a cadlag submartingale with respect to a standard filtration $\{\mathcal{F}_t : 0 \le t \le 1\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose S has property [D]. To prove Theorem <5> we need to construct an increasing linear functional on $\mathcal{H}^+_{\text{BddLip}}$ that is sigma-smooth at 0.

Without loss of generality assume $S_0 \equiv 0$.

Construct μ as a limit of simpler increasing linear functionals on $\mathcal{H}^+_{\text{BddLip}}$. For each n in $\mathbb{N}$ and $i = 0,1,\dots,2^n$ define $t_{i,n} := i/2^n$ and $\Delta_{i,n} := S(t_{i+1,n}) - S(t_{i,n})$ and write $\mathbb{P}_{i,n}(\cdots)$ for expectations conditional on $\mathcal{F}(t_{i,n})$. Note that $\mathbb{P}_{i,n}\Delta_{i,n} \ge 0$ almost surely, by the submartingale property.

For each H in $\mathcal{H}^+_{\text{BddLip}}$, define linear functionals

$$\mu_n H := \sum_{0\le i<2^n} \mathbb{P}\big(H(t_{i,n})\Delta_{i,n}\big) = \sum_{0\le i<2^n} \mathbb{P}\big(H(t_{i,n})\mathbb{P}_{i,n}\Delta_{i,n}\big).$$

The second form ensures that μ_n is an increasing functional on $\mathcal{H}^+_{\text{BddLip}}$.

Existence of the limit

To prove that $\mu H := \lim_{n\to\infty}\mu_n H$ exists for each $H \in \mathcal{H}^+$, I will show that the sequence $\{\mu_n H : n \in \mathbb{N}\}$ is Cauchy. Fix n and m with $n < m$. Define

$$J_i = \{j : t_{i,n} \le t_{j,m} < t_{i+1,n}\}.$$

Then

$$\begin{aligned} \Big|\Big(\sum\nolimits_{j\in J_i} \mathbb{P}H(t_{j,m})\Delta_{j,m}\Big) - \mathbb{P}H(t_{i,n})\Delta_{i,n}\Big| \\ &= \Big|\sum\nolimits_{j\in J_i} \mathbb{P}\big(H(t_{j,m}) - H(t_{i,n})\big)\Delta_{j,m}\Big| \\ &\le \sum\nolimits_{j\in J_i} \mathbb{P}\big(|H(t_{j,m}) - H(t_{i,n})|\mathbb{P}_{j,m}\Delta_{j,m}\big) \\ &\le \sum\nolimits_{j\in J_i} C_H 2^{-n}\mathbb{P}\Delta_{j,m} \\ &= C_H 2^{-n} \sum\nolimits_{j\in J_i} \mathbb{P}\Delta_{i,n} \end{aligned}$$

Sum over i to deduce that $|\mu_m H - \mu_n H| \le C_H 2^{-n}\mathbb{P}S_1$, which tends to zero as n tends to infinity.

A useful upper bound

The functional μ inherits linearity and the increasing property from the $\{\mu_n\}$. For each fixed $\epsilon > 0$, property [D] will give an upper bound for μH in terms of the stopping time

<6> $$\tau(H,\epsilon) := \inf\{t : H(t,\omega) \ge \epsilon\} \wedge 1.$$

Temporarily write τ_n for the discretized stopping time obtained by rounding $\tau(H,\epsilon)$ up to the next integer multiple of 2^{-n}. Then

$$\begin{aligned}
\mu_n H &\le \sum\nolimits_{0\le i<2^n} \mathbb{P}\left(\epsilon\{t_{i,n} < \tau_n\} + C_H\{t_{i,n} \ge \tau_n\}\right)\mathbb{P}_{i,n}\Delta_{i,n}\\
&\le \epsilon \sum\nolimits_{0\le i<2^n} \mathbb{P}\Delta_{i,n} + C_H \sum\nolimits_{0\le i<2^n} \mathbb{P}\{t_{i,n} \ge \tau_n\}\Delta_{i,n}\\
&\le \epsilon\mathbb{P}S_1 + C_H\mathbb{P}\left(S_1 - S_{\tau_n}\right).
\end{aligned}$$

Let n tend to infinity. Uniform integrability of the sequence $\{S_{\tau_n}\}$ together with right-continuity of the sample paths of S lets us deduce in the limit that

<7> $$\mu H \le \epsilon\mathbb{P}S_1 + C_H\mathbb{P}\left(S_1 - S_{\tau(H,\epsilon)}\right).$$

Sigma-smoothness

Now suppose $\{H_k : k \in \mathbb{N}\}$ is a sequence from $\mathcal{H}^+_{\text{BddLip}}$ for which $1 \ge H_k \downarrow 0$ pointwise. For a fixed $\epsilon > 0$, temporarily write σ_k for $\tau(H_k,\epsilon)$. By compactness of $[0,1]$, the pointwise convergence of the continuous functions, $H_k(\cdot,\omega) \downarrow 0$, is actually uniform. For each ω, the sequence $\{\sigma_k(\omega)\}$ not only increases to 1, it actually achieves the value 1 at some finite k (depending on ω). Uniform integrability of $\{S_{\sigma_k} : k \in \mathbb{N}\}$ and the analog of <7> for each H_k then give

$$\mu H_k \le \epsilon\mathbb{P}S_1 + C_H\mathbb{P}\left(S_1 - S_{\sigma_k}\right) \to \epsilon\mathbb{P}S_1 \qquad \text{as } k \to \infty.$$

The sigma-smoothness of μ follows. The functional corresponds to the integral with respect to a finite measure on $\mathcal{P}$, with total mass $\mu[[0,1]] = \lim_n \mu_n[[0,1]] = \mathbb{P}S_1$.

Identification as a Doléans measure

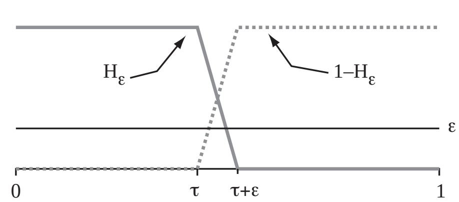
It remains to prove that

(a) $\mu\{0\} \times \Omega = 0$

(b) $\mu[[0,\tau]] = \mathbb{P}S_\tau$ for $\tau \in \mathcal{T}_1$.

Consider first the proof of (b). For given $\epsilon > 0$, approximate $[[0,\tau]]$ by the continuous process

$$H_\epsilon(t,\omega) = \min\left(1, (\tau(\omega) + \epsilon - t)^+/\epsilon\right) \qquad \text{for } 0 \le t \le 1.$$



It is adapted because $\{H_\epsilon(t,\omega) \le c\} = \{\tau \le t - \epsilon(1-c)\} \in \mathcal{F}_t$, for each fixed t and constant $0 \le c < 1$. It belongs to $\mathcal{H}^+_{\text{BddLip}}$, with $C_{H_\epsilon} = 1/\epsilon$, and $[[0,\tau]] \le H_\epsilon \le [[0,\tau+\epsilon]]$.

When $2^{-n} < \epsilon$, direct calculation shows that $\mu_n H_\epsilon \le \mathbb{P}S_{\tau+2\epsilon}$, which in the limit implies $\mu H_\epsilon \le \mathbb{P}S_{\tau+2\epsilon}$. By Dominated Convergence,

$$\mu[[0,\tau]] = \lim_{\epsilon\to 0} \mu H_\epsilon \le \mathbb{P}S_\tau.$$

Inequality <7> applied with $\tau_\epsilon := \tau(1-H_\epsilon, \epsilon)$ gives

$$\mu((\tau+\epsilon, 1]] \le \mu\left(1 - H_\epsilon\right) \le \epsilon\mathbb{P}S_1 + \mathbb{P}\left(S_1 - S_{\tau_\epsilon}\right),$$

which, in the limit as ϵ tends to zero, implies $\mu(\tau, 1]] \le \mathbb{P}\left(S_1 - S_\tau\right)$. From the fact that

$$\mathbb{P}S_1 = \mu[[0,1]] = \mu[[0,\tau]] + \mu((\tau,1]] \le \mathbb{P}S_\tau + \mathbb{P}\left(S_1 - S_\tau\right),$$

conclude that $\mu[[0,\tau]] = \mathbb{P}S_\tau$.

□ Specialize to the case $\tau \equiv 0$ to get (a).

3. Measures as linear functionals

The following material on the Daniell construction of integrals is taken almost verbatim from Pollard (2001, Appendix A), where proofs are given.

<8> **Definition.** *Call a class $\mathcal{H}^+$ of nonnegative real functions on a set $\mathcal{X}$ a* ***lattice cone*** *if it has the following properties. If h, h_1 and h_2 belong to $\mathcal{H}^+$, and α_1 and α_2 are nonegative real numbers, then:*

(H1) $\alpha_1 h_1 + \alpha_2 h_2$ belongs to $\mathcal{H}^+$;

(H2) $h_1 \backslash h_2 := (h_1 - h_2)^+$ belongs to $\mathcal{H}^+$;

(H3) the pointwise minimum $h_1 \wedge h_2$ and maximum $h_1 \vee h_2$ belong to $\mathcal{H}^+$;

(H4) $h \wedge 1$ belongs to $\mathcal{H}^+$.

For a lattice cone $\mathcal{H}^+$, let $\mathcal{K}_0$ denote the class of all sets of the form $K = \{h \ge \alpha\}$, with $h \in \mathcal{H}^+$ and a constant $\alpha > 0$. Notice that $K = \{h' = 1\}$ and $K \le h' \le 1$, where $h' = 1 \wedge (h/\alpha)$. Let $\mathcal{K}$ denote the $\cap c$-closure of $\mathcal{K}_0$. That is, a set K in $\mathcal{K}$ has a representation

<9> $$K = \bigcap\nolimits_{i\in\mathbb{N}} \{h_i \ge \alpha_i\}.$$

The sets in $\mathcal{K}$ are precisely those whose indicator functions are limits of decreasing sequences of functions in $\mathcal{H}^+$. The class $\mathcal{K}$ plays a role similar to that of the compact sets for measures on $\mathcal{B}(\mathbb{R}^k)$. In particular, the class

$$\mathcal{F}(\mathcal{K}) = \{F \subseteq \mathcal{X} : F \cap K \in \mathcal{K} \text{ for all } K \in \mathcal{K}\}$$

has properties analogous to the closed sets, and

$$\mathcal{B}(\mathcal{K}) = \text{sigma-field generated by } \mathcal{F}(\mathcal{K})$$

is analogous to the Borel sigma-field. Each member of $\mathcal{H}^+$ is $\mathcal{B}(\mathcal{K})$-measurable.

<10> **Theorem.** *Let $\mathcal{H}^+$ be a lattice cone, and $T : \mathcal{H}^+ \to \mathbb{R}^+$ be a map for which*

(T1) for nonnegative real numbers α_1, α_2 and functions h_1, h_2 in $\mathcal{H}^+$, $T(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 T h_1 + \alpha_2 T h_2$;

(T2) if $h_1 \le h_2$ pointwise then $Th_1 \le Th_2$.

(T3) $Th_n \downarrow 0$ whenever the sequence $\{h_n\}$ in $\mathcal{H}^+$ decreases pointwise to zero.

(T4) $T(h \wedge n) \to Th$ as $n \to \infty$, for each h in $\mathcal{H}^+$.

Then the set function defined by

$$\mu K := \inf\{Th : K \le h \in \mathcal{H}^+\} \qquad \text{for } K \in \mathcal{K},$$
$$\mu B := \sup\{\mu K : B \supseteq K \in \mathcal{K}\}$$

is a countably additive measure on $\mathcal{B}(\mathcal{K})$ for which $Th = \mu h$ for all h in $\mathcal{H}^+$.

4. Problems

[1] Let $\{X_i : 0 \le i \le n\}$ be a submartingale with $X_0 \equiv 0$. For a fixed $\lambda \in \mathbb{R}^+$, define stopping times $\sigma := \min\{i : X_i \le -\lambda\} \wedge 1$ and $\tau := \min\{i : X_i \ge \lambda\} \wedge 1$.

(i) Show that

$$\lambda \mathbb{P}\{\max_i X_i > \lambda\} \le \mathbb{P}X_\tau\{X_\tau \ge \lambda\} \le \mathbb{P}X_1\{X_\tau \ge \lambda\} \le \mathbb{P}|X_n|.$$

(ii) Show that

$$\lambda \mathbb{P}\{\min_i X_i < -\lambda\} \le \mathbb{P}(-X_\sigma)\{X_\sigma \le -\lambda\}$$
$$\le -\mathbb{P}X_\sigma + \mathbb{P}X_n\{X_\sigma > -\lambda\} \le \mathbb{P}|X_n|.$$

(iii) Suppose $\{Y_t : 0 \le t \le 1\}$ is a cadlag submartingale with $Y_0 \equiv 0$. Show that $\lambda \mathbb{P}\{\sup_t |Y_t| > \lambda\} \le 2\mathbb{P}|Y_1|$.

[2] Suppose a cadlag martingale $\{S_t : 0 \le t \le 1\}$, with $S_0 \equiv 0$, has a Doléans measure μ in the sense of Definition <2>, that is, $\mu((0, \tau]] = \mathbb{P}S_\tau$ for every $\tau \in \mathcal{T}_1$. Show that S has property [D] by following these steps.

(i) For a given $\tau \in \mathcal{T}_1$, let τ_n be the stopping time obtained by rounding up to the next integer multiple of 2^{-n}.

(ii) Invoke the Stopping Time Lemma to show that $0 \le \mathbb{P}S_{\tau_n}$ and $\mathbb{P}S_{\tau_n}^+ \le \mathbb{P}S_1^+$ for each $\tau \in \mathcal{T}_1$. Deduce that $\mathbb{P}|S_{\tau_n}| \le \kappa := 2\mathbb{P}S_1^+ < \infty$.

(iii) Invoke Fatou's lemma to show that $\sup_{\tau \in \mathcal{T}_1} \mathbb{P}|S_\tau| \le \kappa$.

(iv) For each $C \in \mathbb{R}^+$, show that

$$\mathbb{P}S_{\tau_n}\{S_{\tau_n} > C\} \le \mathbb{P}S_1\{S_{\tau_n} > C\} \le \mathbb{P}S_1\{S_1 > \sqrt{C}\} + \kappa/\sqrt{C}.$$

Invoke Fatou, then deduce that $\sup_{\tau \in \mathcal{T}_1} \mathbb{P}S_\tau\{S_\tau > C\} \to 0$ as $C \to \infty$.

(v) Show that every cadlag function on $[0, 1]$ is bounded in absolute value. Deduce that the stopping time $\sigma_C := \inf\{t : S_t < -C\} \wedge 1$ has $\sigma_C(\omega) = 1$ for all C large enough (depending on ω). Deduce that $\mu((\sigma_C, 1]] \to 0$ as $C \to \infty$.

(vi) For a given $\tau \in \mathcal{T}_1$ and $C \in \mathbb{R}^+$, define $F_\tau := \{S_\tau < -C\}$. Show that $\tau' := \tau F_\tau^c + F_\tau$ is a stopping time for which

$$\mathbb{P}\left(S_1 - S_\tau\right) F_\tau = \mathbb{P}\left(S_{\tau'} - S_\tau\right) = \mu((\tau, \tau']] \leq \mu((\sigma_C, 1]],$$

Hint: Show that if $\omega \in F_\tau$ then $\sigma_C(\omega) \leq \tau(\omega)$ and if $\omega \in F_\tau^c$ then $\tau(\omega) = \tau'(\omega)$.

(vii) Deduce that $\sup_{\tau \in \mathcal{T}_1} \mathbb{P}\left(-S_\tau\right)\{S_\tau < -C\} \to 0$ as $C \to \infty$.

[3] Let $\{S_t : t \in \mathbb{R}^+\}$ be a submartingale of class [D]. Show that there exists an integrable random variable S_∞ for which $\mathbb{P}\left(S_\infty \mid \mathcal{F}_t\right) \geq S_t \to S_\infty$ almost surely and in L^1 by following these steps.

(i) Show that the uniformly integrable submartingale $\{S_n : n \in \mathbb{N}\}$ converges almost surely and in L^1 to an S_∞ for which $\mathbb{P}\left(S_\infty \mid \mathcal{F}_n\right) \geq S_n$.

(ii) For $t \leq n$, show that $S_t \leq \mathbb{P}\left(\mathbb{P}(S_\infty \mid \mathcal{F}_n) \mid \mathcal{F}_t\right) = \mathbb{P}\left(S_\infty \mid \mathcal{F}_t\right)$.

(iii) For $t \geq n$, show that

$$\mathbb{P}(S_t - S_n)^- \leq \mathbb{P}(S_t - S_n)^+ \leq \mathbb{P}(S_\infty - S_n)^+ \to 0 \qquad \text{as } n \to \infty.$$

(iv) For each $k \in \mathbb{N}$, choose $n(k)$ for which $\mathbb{P}|S_\infty - S_{n(k)}| \leq 4^{-k}$. Invoke Problem [1] to show that $\sum_k \mathbb{P}\{\sup_{t \geq n(k)} |S_t - S_{n(k)}| > 2^{-k}\} < \infty$.

(v) Deduce that $S_t \to S_\infty$ almost surely.

5. Notes

My exposition in this Chapter is based on ideas drawn from a study of Métivier (1982, §13), Dellacherie & Meyer (1982, Chapter VII), and Chung & Williams (1990, Chapter 2). The construction in Section 2 appears new, although it is clearly closely related to existing methods.

References

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