Analytic sets

For a discrete-time process $\{X_n\}$ adapted to a filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$, the prime example of a stopping time is $\tau = \inf\{n \in \mathbb{N} : X_n \in B\}$, the first time the process enters some Borel set B. For a continuous-time process $\{X_t\}$ adapted to a filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$, it is less obvious whether the analogously defined random variable $\tau = \inf\{t : X_t \in B\}$ is a stopping time. (Also it is not necessarily true that X_τ is a point of B.) The most satisfactory resolution of the underlying measure-theoretic problem requires some theory about analytic sets. What follows is adapted from Dellacherie & Meyer (1978, Chapter III, paras 1–33, 44–45). The following key result will be proved in this handout.

<1> **Theorem.** *Let A be a $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$-measurable subset of $\mathbb{R}^+ \times \Omega$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Then:*

(i) The projection $\pi_\Omega A := \{\omega \in \Omega : (t, \omega) \in A$ for some t in $\mathbb{R}^+\}$ belongs to $\mathcal{F}$.

(ii) There exists an $\mathcal{F}$-measurable random variable $\psi : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ such that $\psi(\omega) < \infty$ and $(\psi(\omega), \omega) \in A$ for almost all ω in the projection $\pi_\Omega A$, and $\psi(\omega) = \infty$ for $\omega \notin \pi_\Omega A$.

Remark. The map ψ in (ii) is called a ***measurable cross-section*** of the set A. Note that the cross-section $A_\omega := \{t \in \mathbb{R}^+ : (t, \omega) \in A\}$ is empty when $\omega \notin \pi_\Omega A$. It would be impossible to have $(\psi(\omega), \omega) \in A$ for such an ω.

The proofs will exploit the properties of the collection of analytic subsets of $[0, \infty] \times \Omega$. As you will see, the analytic sets have properties analogous to those of sigma-fields—stability under the formation of countable unions and intersections. They are not necessarily stable under complements, but they do have an extra stability property for projections that is not shared by measurable sets. The Theorem is made possible by the fact that the product-measurable subsets of $\mathbb{R}^+ \times \Omega$ are all analytic.

1. Notation

A collection $\mathcal{D}$ of subsets of a set $\mathcal{X}$ with $\emptyset \in \mathcal{D}$ is called a ***paving*** on $\mathcal{X}$. A paving that is closed under the formation of unions of countable subcollections is said to be a $\cup c$-paving. For example, the set $\mathcal{D}_\sigma$ of all unions of countable subcollections of $\mathcal{D}$ is a $\cup c$-paving. Similarly, the set $\mathcal{D}_\delta$ of all intersections of countable subcollections of $\mathcal{D}$ is a $\cap c$-paving. Note that $\mathcal{D}_{\sigma\delta} := (\mathcal{D}_\sigma)_\delta$ is a $\cap c$-paving but it need not be stable under $\cup c$.

Let T be a compact metric space equipped with the paving $\mathcal{K}(T)$ of compact subsets and its Borel sigma-field $\mathcal{B}(T)$, which is generated by $\mathcal{K}(T)$.

Remark. In fact, $\mathcal{K}(T)$ is also the class of closed subsets of the compact T.

For Theorem <1>, the appropriate space will be $T = [0, \infty]$. The sets in $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$ can be identified with sets in $\mathcal{B}(T) \otimes \mathcal{F}$. The compactness of T will be needed to derive good properties for the projection map $\pi_\Omega : T \times \Omega \to \Omega$.

An important role will be played by the $\cap f$-paving

$$\mathcal{K}(T) \times \mathcal{F} := \{K \times F : K \in \mathcal{K}(T),\ F \in \mathcal{F}\} \qquad \text{on } T \times \Omega$$

and by the paving $\mathcal{R}$ that consists of all finite unions of sets from $\mathcal{K}(T) \times \mathcal{F}$. That is, $\mathcal{R}$ is the $\cup f$-closure of $\mathcal{K}(T) \times \mathcal{F}$. Note (Problem [1]) that $\mathcal{R}$ is a

$(\cup f, \cap f)$-paving on $T \times \Omega$. Also, if $R = \cup_i K_i \times F_i$ then, assuming we have discarded any terms for which $K_i = \emptyset$,

$$\pi_\Omega(R) = \cup_i \pi_\Omega \left(K_i \times F_i\right) = \cup_i F_i \in \mathcal{F}.$$

REMARK. If $\mathcal{E}$ and $\mathcal{F}$ are sigma-fields, note the distinction between

$$\mathcal{E} \times \mathcal{F} = \{E \times F : E \in \mathcal{E},\ F \in \mathcal{F}\}$$

and $\mathcal{E} \otimes \mathcal{F} := \sigma(\mathcal{E} \times \mathcal{F})$.

2. Why compact sets are needed

Many of the measurability difficulties regarding projections stem from the fact that they do not "preserve set-theoretic operations" in the way that inverse images do: $\pi_\Omega\left(\cup_i A_i\right) = \cup_i \pi_\Omega A_i$ but $\pi_\Omega\left(\cap_i A_i\right) \subseteq \cap_i \pi_\Omega A_i$. Compactness of cross-sections will allow us to strengthen the last inclusion to an equality.

<2> **Lemma.** *[Finite intersection property] Suppose $\mathcal{K}_0$ is a collection of compact subsets of a metric space $\mathcal{X}$ for which each finite subcollection has a nonempty intersection. Then $\cap\mathcal{K}_0 \neq \emptyset$.*

also works for any Hausdorff topological space

Proof. Arbitrarily choose a K_0 from $\mathcal{K}_0$. If $\cap\mathcal{K}_0$ were empty then the sets $\{K^c : K \in \mathcal{K}_0\}$ would be an open cover of K_0. Extract a finite subcover $\cup_{i=1}^m K_i^c$. Then $\cap_{i=0}^m K_i = \emptyset$, a contradiction. □

<3> **Corollary.** *Suppose $\{A_i : i \in \mathbb{N}\}$ is a decreasing sequence of subsets of $T \times \Omega$ for which each ω-cross-section $K_i(\omega) := \{t \in T : (t, \omega) \in A_i\}$ is compact. Then $\pi_\Omega\left(\cap_{i\in\mathbb{N}} A_i\right) = \cap_{i\in\mathbb{N}} \pi_\Omega A_i$.*

Proof. Suppose $\omega \in \cap_{i\in\mathbb{N}} \pi_\Omega A_i$. Then $\{K_i(\omega) : i \in \mathbb{N}\}$ is a decreasing sequence of compact, nonempty (because $\omega \in \pi_\Omega A_i$) subsets of T. The finite intersection property of compact sets ensures that there is a t in $\cap_{i\in\mathbb{N}} K_i(\omega)$. The point (t, ω) belongs to $\cap_{i\in\mathbb{N}} A_i$ and $\omega \in \pi_\Omega\left(\cap_i A_i\right)$. □

REMARK. For our applications, we will be dealing only with sequences, but the argument also works for more general collections of sets with compact cross-sections.

<4> **Corollary.** *If $B = \cap_{i\in\mathbb{N}} R_i$ with $R_i \in \mathcal{R}$ then $\pi_\Omega B = \cap_{i\in\mathbb{N}} \pi_\Omega R_i \in \mathcal{F}$.*

Proof. Note that the cross-section of each $\mathcal{R}$-set is a finite union of compact sets, which is compact. Without loss of generality, we may assume that $R_1 \supseteq R_2 \supseteq \ldots$. Invoke Corollary <3>. □

3. Measurability of some projections

For which $B \in \mathcal{B}(T) \otimes \mathcal{F}$ is it true that $\pi_\Omega(B) \in \mathcal{F}$? From Corollary <4>, we know that it is true if B belongs to $\mathcal{R}_\delta$. The following properties of outer measures (see Problem [2]) will allow us to extend this nice behavior to sets in $\mathcal{R}_{\sigma\delta}$:

(i) If $A_1 \subseteq A_2$ then $\mathbb{P}^*(A_1) \le \mathbb{P}^*(A_2)$

(ii) If $\{A_i : i \in \mathbb{N}\}$ is an increasing sequence then $\mathbb{P}^*\left(A_i\right) \uparrow \mathbb{P}^*\left(\cup_{i\in\mathbb{N}} A_i\right)$.

(iii) If $\{F_i : i \in \mathbb{N}\} \subseteq \mathcal{F}$ is a decreasing sequence then

$$\mathbb{P}^*\left(F_i\right) = \mathbb{P}F_i \downarrow \mathbb{P}\left(\cap_{i\in\mathbb{N}} F_i\right) = \mathbb{P}^*\left(\cap_{i\in\mathbb{N}} F_i\right).$$

For each subset D of $T \times \Omega$ define $\Psi^*(D) := \mathbb{P}^*\pi_\Omega D$, the outer measure of the projection of D onto Ω. If $D_i \uparrow D$ then $\pi_\Omega D_i \uparrow \pi_\Omega D$. If $R_i \in \mathcal{R}$ and

$R_i \downarrow B$ then $\pi_\Omega R_i \in \mathcal{F}$ and $\pi_\Omega R_i \downarrow \pi_\Omega B \in \mathcal{F}$. The properties for $\mathbb{P}^*$ carry over to analogous properties for Ψ^*:

(i) If $D_1 \subseteq D_2$ then $\Psi^*(D_1) \le \Psi^*(D_2)$

(ii) If $\{D_i : i \in \mathbb{N}\}$ is an increasing sequence then $\Psi^*\left(D_i\right) \uparrow \Psi^*\left(\cup_{i\in\mathbb{N}} D_i\right)$.

(iii) If $\{R_i : i \in \mathbb{N}\} \subseteq \mathcal{R}$ is a decreasing sequence then $\Psi^*\left(R_i\right) \downarrow \Psi^*\left(\cap_{i\in\mathbb{N}} R_i\right)$.

With just these properties, we can show that π_Ω behaves well on a much larger collection of sets than $\mathcal{R}$.

<5> **Lemma.** *If $A \in \mathcal{R}_{\sigma\delta}$ then $\Psi^*(B) = \sup\{\Psi^*(B) : B \in \mathcal{R}_\delta\}$. Consequently, the set $\pi_\Omega A$ belongs to $\mathcal{F}$.*

Proof. Write A as $\cap_{i\in\mathbb{N}} D_i$ with $D_i = \cup_{j\in\mathbb{N}} R_{ij} \in \mathcal{R}_\sigma$. As $\mathcal{R}$ is $\cup f$-stable, we may assume that R_{ij} is increasing in j for each fixed i.

Suppose $\Psi^*(A) > M$ for some constant M. Invoke (ii) for the sequence $\{AR_{1j}\}$, which increases to $AD_1 = A$, to find an index j_1 for which the set $R_1 := R_{1j_1}$ has $\Psi^*\left(AR_1\right) > M$.

The sequence $\{AR_1R_{2j}\}$ increases to $AR_1D_2 = AR_1$. Again by (ii), there exists an index j_2 for which the set $R_2 = R_{2j_2}$ has $\Psi^*(AR_1R_2) > M$. And so on. In this way we construct sets R_i in $\mathcal{R}$ for which

$$\Psi^*(R_1R_2\dots R_n) \ge \Psi^*(AR_1R_2\dots R_n) > M$$

for every n. The set $B_M := \cap_{i\in\mathbb{N}} R_i$ belongs to $\mathcal{R}_\delta$; it is a subset of $\cap_{i\in\mathbb{N}} D_i = A$; and, by (iii), $\Psi^*(B) \ge M$.

By Corollary <4>, the set B_M projects to a set $F_M := \pi_\Omega B_M$ in $\mathcal{F}$ and hence $\mathbb{P}F_M = \Psi^* B \ge M$. The set $\pi_\Omega A$ is inner regular, in the sense that

$$\mathbb{P}^*\pi_\Omega A = \Psi^* A = \sup\{\mathbb{P}F : \pi_\Omega A \supseteq F \in \mathcal{F}\}$$

□ It follows (Problem [2]) that the set $\pi_\Omega A$ belongs to $\mathcal{F}$.

The properties shared by $\mathbb{P}^*$ and Ψ^* are so useful that they are given a name.

<6> **Definition.** *Suppose $\mathcal{S}$ is a paving on a set S. A function Ψ defined for all subsets of S and taking values in $[-\infty, \infty]$ is said to be a **Choquet $\mathcal{S}$-capacity** if it satisfies the following three properties.*

(i) If $D_1 \subseteq D_2$ then $\Psi(D_1) \le \Psi(D_2)$

(ii) If $\{D_i : i \in \mathbb{N}\}$ is an increasing sequence then $\Psi\left(D_i\right) \uparrow \Psi\left(\cup_{i\in\mathbb{N}} D_i\right)$.

(iii) If $\{S_i : i \in \mathbb{N}\} \subseteq \mathcal{S}$ is a decreasing sequence then $\Psi\left(S_i\right) \downarrow \Psi\left(\cap_{i\in\mathbb{N}} S_i\right)$.

The outer measure $\mathbb{P}^*$ is a Choquet $\mathcal{F}$-capacity defined for the subsets of Ω. Moreover, if Ψ is any Choquet $\mathcal{F}$-capacity defined for the subsets of Ω then $\Psi^*(D) := \Psi(\pi_\Omega D)$ is a Choquet $\mathcal{R}$-capacity defined for the subsets of $T \times \Omega$. The argument from Lemma <5> essentially shows that if $A \in \mathcal{R}_{\sigma\delta}$ then $\Psi^*(B) = \sup\{\Psi^*(B) : B \in \mathcal{R}_\delta\}$ for every such Ψ^*, whether defined via $\mathbb{P}^*$ or not.

4. Analytic sets

The paving of $\mathcal{S}$-analytic sets can be defined for any paving $\mathcal{S}$ on a set S. For our purposes, the most important case will be $S = T \times \Omega$ with $\mathcal{S} = \mathcal{R}$.

<7> **Definition.** *Suppose $\mathcal{S}$ is a paving on a set S. A subset A of S is said to be $\mathcal{S}$-analytic if there exists a compact metric space E and a subset D in $\left(\mathcal{K}(E) \times \mathcal{S}\right)_{\sigma\delta}$ for which $A = \pi_S D$. Write $\mathcal{A}(\mathcal{S})$ for the set of all $\mathcal{S}$-analytic subsets of S.*

REMARK. Note that $\mathcal{R}_{\sigma\delta} = (\mathcal{K}(T) \times \mathcal{F})_{\sigma\delta}$. The σ takes care of the $\cup f$ operation needed to generate $\mathcal{R}$ from $\mathcal{K}(T) \times \mathcal{F}$. The $\mathcal{R}$-analytic sets are also called $\mathcal{K}(T) \times \mathcal{F}$-analytic sets.

In fact, it is possible to find a single E that defines all the $\mathcal{S}$-analytic subsets, but that possibility is not important for my purposes. What is important is the fact that $\mathcal{A}(\mathcal{S})$ is a $(\cup c, \cap c)$-paving: see Problem [3].

When E is another compact metric space, Tychonoff's theorem (see Dudley 1989, Section 2.2, for example) ensures not only that the product space $E \times T$ is a compact metric space but also that $\mathcal{K}(E) \times \mathcal{K}(T) \subseteq \mathcal{K}(E \times T)$.

Lemma <5>, applied to $\widetilde{T} := E \times T$ instead of T and with $\widetilde{\mathcal{R}}$ the $\cup f$-closure of $\mathcal{K}(E \times T) \times \mathcal{F}$, implies that

<8> $$\widetilde{\pi}_\Omega D \in \mathcal{F} \qquad \text{for each } D \text{ in } \widetilde{\mathcal{R}}_{\sigma\delta}.$$

Here $\widetilde{\pi}_\Omega$ projects $E \times T \times \Omega$ onto Ω. We also have

$$\widetilde{\mathcal{R}}_{\sigma\delta} \supseteq \left(\mathcal{K}(E) \times \mathcal{K}(T) \times \mathcal{F}\right)_{\sigma\delta} = \left(\mathcal{K}(E) \times \mathcal{R}\right)_{\sigma\delta}$$

where $\mathcal{R}$ is the $\cup f$-closure of $\mathcal{K}(T) \times \mathcal{F}$, as in Section 3. As a special case of property <8> we have

<9> $$\widetilde{\pi}_\Omega D \in \mathcal{F} \qquad \text{for each } D \text{ in } \left(\mathcal{K}(E) \times \mathcal{R}\right)_{\sigma\delta}.$$

Write $\widetilde{\pi}_\Omega$ as a composition of projection $\pi_\Omega \circ \widetilde{\pi}_{T\times\Omega}$, where $\widetilde{\pi}_{T\times\Omega}$ projects $E \times T \times \Omega$ onto $T \times \Omega$. As E ranges over all compact metric spaces and D ranges over all the $\left(\mathcal{K}(E) \times \mathcal{R}\right)_{\sigma\delta}$ sets, the projections $A := \widetilde{\pi}_{T\times\Omega} D$ range over all $\mathcal{R}$-analytic subsets of $T \times \Omega$. Property <9> is equivalent to the assertion

<10> $$\pi_\Omega A \in \mathcal{F} \qquad \text{for all } A \in \mathcal{A}(\mathcal{R}).$$

In fact, the method used to prove Lemma <5> together with an analogue of the argument just outlined establishes an approximation theorem for analytic sets and general Choquet capacities.

<11> **Theorem.** *Suppose $\mathcal{S}$ is a $(\cup f, \cap f)$-paving on a set S and Let Ψ is a Choquet $\mathcal{S}$-capacity on S. Then $\Psi(A) = \sup\{\Psi(B) : A \supseteq B \in \mathcal{S}_\delta\}$. for each A in $\mathcal{A}(\mathcal{S})$.*

To prove assertion (i) of Theorem <1>, we have only to check that

$$\mathcal{B}(T) \otimes \mathcal{F} \subseteq \mathcal{A}(\mathcal{R})$$

for the special case where $T = [0, \infty]$. By Problem [3], $\mathcal{A}(\mathcal{R})$ is a $(\cup c, \cap c)$-paving. It follows easily that

$$\mathcal{H} := \{H \in \mathcal{B}(T) \otimes \mathcal{F} : H \in \mathcal{A}(\mathcal{R}) \text{ and } H^c \in \mathcal{A}(\mathcal{R})\}$$

is a sigma-field on $T \times \Omega$. Each $K \times F$ with $K \in \mathcal{K}(T)$ and $F \in \mathcal{F}$ belongs to $\mathcal{H}$ because $\mathcal{K}(T) \times \mathcal{F} \subseteq \mathcal{R} \subseteq \mathcal{A}(\mathcal{R})$ and

$$(K \times F)^c = \left(K \times F^c\right) + \left(K^c \times \Omega\right)$$
$$K^c = \cup_{i\in\mathbb{N}}\{t : d(t, K) \ge 1/i\} \in \mathcal{K}(T)_\sigma$$

It follows that $\mathcal{H} = \sigma(\mathcal{K}(T) \times \mathcal{F}) = \mathcal{B}(T) \otimes \mathcal{F}$ and $\mathcal{B}(T) \otimes \mathcal{F} \subseteq \mathcal{A}(\mathcal{R})$.

5. Existence of measurable cross-sections

The general Theorem <11> is exactly what we need to prove part (ii) of Theorem <1>.

Once again identify A with an $\mathcal{R}$-analytic subset of $T \times \Omega$, where $T = [0, \infty]$. The result is trivial if $\alpha_1 := \mathbb{P}\pi_\Omega A = 0$, so assume $\alpha_1 > 0$.

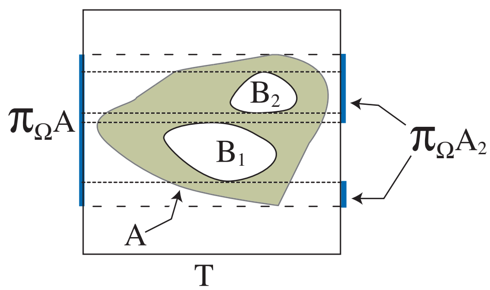
Invoke Theorem <11> for the $\mathcal{R}$-capacity defined by $\Psi^*(D) = \mathbb{P}^*(\pi_\Omega D)$. Find a subset with $A \supseteq B_1 \in \mathcal{R}_\delta$ and $\mathbb{P}\left(\pi_\Omega B_1\right) = \Psi^*(B_1) \geq \alpha_1/2$. Define

$$\psi_1(\omega) := \inf\{t \in \mathbb{R}^+ : (t, \omega) \in B_1\}.$$

Because the set B_1 has compact cross-sections, the infimum is actually achieved for each ω in $\pi_\Omega B_1$. For $\omega \notin \pi_\Omega B_1$ the infimum equals ∞. Define

$$A_2 := \{(t, \omega) \in A : \omega \notin \pi_\Omega B_1\} = A \cap \left(T \times (\pi_\Omega B_1)^c\right)$$

Note that $A_2 \in \mathcal{A}(\mathcal{R})$ and $\alpha_2 := \mathbb{P}\pi_\Omega A_2 \leq \alpha_1/2$. Without loss of generality suppose $\alpha_2 > 0$. Find a subset with $A_2 \supseteq B_2 \in \mathcal{R}_\delta$ and $\mathbb{P}\left(\pi_\Omega B_2\right) = \Psi^*(B_2) \geq \alpha_2/2$. Define $\psi_2(\omega)$ as the first hitting time on B_2.



And so on. The sets $\{\pi_\Omega B_i : i \in \mathbb{N}\}$ are disjoint, with $F := \cup_{i\in\mathbb{N}}\pi_\Omega B_i$ a subset of $\pi_\Omega A$. By construction $\alpha_i \downarrow 0$, which ensures that $\mathbb{P}\left((\pi_\Omega A)\backslash F\right) = 0$.

If $\alpha_i = 0$ for some i, the construction requires only finitely many steps.

Define $\psi := \inf_{i\in\mathbb{N}} \psi_i$. On B we have $(\psi(\omega), \omega) \in A$. □

6. Problems

[1] Suppose $\mathcal{S}$ is a paving (on a set S), which is $\cap f$-stable. Let $\mathcal{S}_{\cup f}$ consists of the set of all unions of finite collections of sets from $\mathcal{S}$. Show that $\mathcal{S}_{\cup f}$ is a $(\cup f, \cap f)$-paving. Hint: Show that $(\cup_i S_i) \cap (\cup_j T_j) = \cup_{i,j}(S_i \cap T_j)$.

[2] The outer measure of a set $A \subseteq \Omega$ is defined as $\mathbb{P}A := \inf\{\mathbb{P}F : A \subseteq F \in \mathcal{F}\}$.

(i) Show that the infimum is achieved, that is, there exists an $F \in \mathcal{F}$ for which $A \subseteq F$ and $\mathbb{P}^*A = \mathbb{P}F$. Hint: Consider the intersection of a sequence of sets for which $\mathbb{P}F_n \downarrow \mathbb{P}^*A$.

(ii) Suppose $\{D_n : n \in \mathbb{N}\}$ is an increasing sequence of sets (not necessarily $\mathcal{F}$-measurable) with union D. Show that $\mathbb{P}^*D_n \uparrow \mathbb{P}^*D$. Hint: Find sets with $D_i \subseteq F_i \in \mathcal{F}$ and $\mathbb{P}^*D_i = \mathbb{P}F_i$. Show that $\cap_{i\geq n}F_i \uparrow F \supseteq D$ and $\mathbb{P}F \leq \sup_{i\in\mathbb{N}} \mathbb{P}^*D_i$.

(iii) Suppose D is a subset of Ω for which $\mathbb{P}^*D = \sup\{\mathbb{P}F_0 : D \supseteq F_0 \in \mathcal{F}\}$. Show that D belongs to the $\mathbb{P}$-completion of $\mathcal{F}$ (or to $\mathcal{F}$ itself if $\mathcal{F}$ is $\mathbb{P}$-complete). Hint: Find sets F and F_i in $\mathcal{F}$ for which $F_i \subseteq D \subseteq F$ and $\mathbb{P}F_i \uparrow \mathbb{P}^*D = \mathbb{P}F$. Show that $F\backslash\left(\cup_{i\in\mathbb{N}} F_i\right)$ has zero $\mathbb{P}$-measure.

[3] Suppose $\{A_\alpha : \alpha \in \mathbb{N}\} \subseteq \mathcal{A}(\mathcal{S})$. Show that $\cup_\alpha A_a \in \mathcal{A}(\mathcal{S})$ and $\cap_\alpha A_a \in \mathcal{A}(\mathcal{S})$, by the following steps. Recall that there exist compact metric spaces $\{E_\alpha : \alpha \in \mathbb{N}\}$, each equipped with its paving $\mathcal{K}_\alpha$ of compact subsets, and sets $D_\alpha \in \left(\mathcal{K}_\alpha \times \mathcal{S}\right)_{\sigma\delta}$ for which $A_\alpha = \pi_S D_\alpha$.

D&M Theorem 3.8

(i) Define $E := \times_{\alpha\in\mathbb{N}} E_\alpha$ and $E_{-\beta} = \times_{\alpha\in\mathbb{N}\setminus\{\beta\}} E_\alpha$. Show that E is a compact metric space.

(ii) Define $\widetilde{D} := D_\alpha \times E_{-\alpha}$. Show that $\widetilde{D}_\alpha \in \big(\mathcal{K}(E) \times \mathcal{S}\big)_{\sigma\delta}$ and that $A_\alpha = \widetilde{\pi}_S \widetilde{D}_\alpha$, where $\widetilde{\pi}_S$ denotes the projection map from $E \times S$ to S.

(iii) Show that $\cap_\alpha A_\alpha = \widetilde{\pi}_S \left(\cap_\alpha \widetilde{D}_\alpha \right)$ and $\cap_\alpha \widetilde{D}_\alpha \in \big(\mathcal{K}(E) \times \mathcal{S}\big)_{\sigma\delta}$.

(iv) Without loss of generality suppose the E_α spaces are disjoint—otherwise replace E_α by $\{\alpha\} \times E_\alpha$. Define $H = \cup_{\alpha\in\mathbb{N}} E_\alpha$ and $E^* := H \cup \{\infty\}$. Without loss of generality suppose the metric d_α on E_α is bounded by $2^{-\alpha}$. Define

$$d(x, y) = d(y, x) := \begin{cases} d_\alpha(x, y) & \text{if } x, y \in E_\alpha \\ 2^{-\alpha} + 2^{-\beta} & \text{if } x \in E_\alpha,\ y \in E_\beta \text{ with } \alpha \neq \beta \\ 2^{-\alpha} & \text{if } y = \infty \text{ and } x \in E_\alpha \end{cases}$$

Show that E^* is a compact metric space under d.

(v) Suppose $D_\alpha = \cap_{i\in\mathbb{N}} B_{\alpha i}$ with $B_{\alpha i} \in (\mathcal{K}_\alpha \times \mathcal{S})_\sigma$. Show that $\cup_\alpha D_\alpha = \cap_i \cup_\alpha B_{\alpha,i}$. Hint: Consider the intersection with $E_\alpha \times S$.

(vi) Deduce that $\cup_\alpha D_\alpha \in (\mathcal{K}(E^*) \times \mathcal{S})_{\sigma\delta}$.

(vii) Conclude that $\cup_\alpha A_\alpha = \pi_S \cup_\alpha D_\alpha \in \mathcal{A}(\mathcal{S})$.

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