Project 1

I suggest that you bring a copy of this sheet to the Friday session and make rough notes on it while I explain some of the ideas. You should then go home and write a reasonably self-contained account in your notebook. You may consult any texts you wish and you may ask me or anyone else as many questions as you like.

Please do not just copy out standard proofs without understanding. Please do not just copy from someone else's notebook.

In your weekly session—DON'T FORGET TO ARRANGE A TIME WITH ME—*I will discuss with you any difficulties you have with producing an account in your own words. I will also point out refinements, if you are interested.*

At the end of the semester, I will look at your notebook to make up a grade. By that time, you should have a pretty good written account of a significant chunk of stochastic calculus.

Things to explain in your notebook:

(i) filtrations and stochastic processes adapted to a filtration
(ii) stopping times and related sigma-fields
(iii) (sub/super)martingales in continuous time
(iv) How does progressive measurability help?
(v) cadlag sample paths
(vi) versions of stochastic processes
(vii) standard filtrations: Why are they convenient?
(viii) cadlag versions of martingales adapted to standard filtrations

Please pardon my grammar. This sheet is witten in note form, not in real sentences.

Filtrations and stochastic processes

Fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Negligible sets $\mathcal{N} := \{N \in \mathcal{F} : \mathbb{P}N = 0\}$. Without loss of generality the probability space is complete.

completeness needed later

Time set $T \subseteq \mathbb{R} \cup -\infty \cup \{\infty\}$. Filtration $\{\mathcal{F}_t : t \in T\}$: set of sub-sigma-fields of $\mathcal{F}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s < t$. Think of $\mathcal{F}_t$ as "information available at time t"?

If $\infty \notin T$ define $\mathcal{F}_\infty := \sigma\left(\cup_{t\in T} \mathcal{F}_t\right)$.

Stochastic process $\{X_t : t \in T\}$: a set of $\mathcal{F}$-measurable random variables. Write $X_t(\omega)$ or $X(t,\omega)$. We can think of X as a map from $T \times \Omega$ into $\mathbb{R}$.

Say that X is ***adapted to the filtration*** if X_t is $\mathcal{F}_t$-measurable for each $t \in T$. Value of $X_t(\omega)$ can be determined by the "information available at time t".

Stopping times

Function $\tau : \Omega \to \overline{T} := T \cup \{\infty\}$ such that $\{\omega : \tau(\omega) \le t\} \in \mathcal{F}_t$ for each $t \in T$. Check that τ is $\mathcal{F}_\infty$-measurable. Define

$$\mathcal{F}_\tau := \{F \in \mathcal{F}_\infty : F\{\tau \le t\} \in \mathcal{F}_t \text{ for each } t \in T\}$$

Check that $\mathcal{F}_\tau$ is a sigma-field. Show that τ is $\mathcal{F}_\tau$-measurable. Show that an $\mathcal{F}_\infty$-measurable random variable Z is $\mathcal{F}_\tau$-measurable if and only if $Z\{\tau \le t\}$ is $\mathcal{F}_t$-measurable for each $t \in T$.

Progressive measurability

Problem: If $\{X_t : t \in T\}$ is adapted and τ is a stopping time, when is the function

$$\omega \mapsto X(\tau(\omega),\omega)\{\tau(\omega) < \infty\}$$

$\mathcal{F}\tau$-measurable? Perhaps simplest to think only of the case where $T = \mathbb{R}^+$, equipped with its Borel sigma-field $\mathcal{B}(T)$.

Warmup: Suppose τ takes values in T. Show $X(\tau(\omega),\omega)$ is $\mathcal{F}$-measurable.

$$\begin{array}{ccccc} \omega & \mapsto & (\tau(\omega),\omega) & \mapsto & X(\tau(\omega),\omega) \\ \Omega & & T \times \Omega & & \mathbb{R} \\ \mathcal{F} & & \mathcal{B}(T)\otimes\mathcal{F} & & \mathcal{B}(\mathbb{R}) \end{array}$$

If X is $\mathcal{B}(T)\otimes\mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurable, we get $\mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurability for the composition.

Abbreviate $\mathcal{B}([0,t])$, the Borel sigma-field on $[0,t]$, to $\mathcal{B}_t$.

- Now suppose X is ***progressively measurable***, that is, the restriction of X to $[0,t]\times\Omega$ is $\mathcal{B}_t\otimes\mathcal{F}_t$-measurable for each $t \in T$. For a fixed t, write Y for the restriction of X to $[0,t]\times\Omega$. Show that

  $$X(\tau(\omega),\omega)\{\tau(\omega) \le t\} = Y(\tau(\omega)\wedge t,\omega)\{\tau(\omega) \le t\}$$

  Adapt the warmup argument to prove that $Y(\tau(\omega)\wedge t,\omega)$ is $\mathcal{F}_\tau$-measurable. Then what? Conclude that $X(\tau(\omega),\omega)\{\tau(\omega) < \infty\}$ is $\mathcal{F}_\tau$-measurable.

- Show that an adapted process with right-continuous sample paths is progressively measurable. Argue as follows, for a fixed t. Define $t_i = it/n$ and

  $$X_n(s,\omega) = X(0,\omega) + \sum_{i\le n} X(t_i,\omega)\{t_{i-1} < s \le t_i\} \qquad \text{for } 0 \le s \le t.$$

  Show that X_n is $\mathcal{B}_t\otimes\mathcal{F}_t$-measurable and X_n converges pointwise to the restriction of X to $[0,t]\times\Omega$.

Cadlag

Define $\mathbb{D}(T)$ as the set of real valued functions on T that are right continuous and have left limits at each t. (Modify the requirements suitably at points not in the interior of T.) Say that functions in $\mathbb{D}(T)$ are cadlag on T.

Say that a process X has ***cadlag sample paths*** if the function $t \mapsto X(t,\omega)$ is cadlag for each fixed ω.

A typical sample path problem

For a fixed integable random variable ξ, define $X_t(\omega) = \mathbb{P}(\xi \mid \mathcal{F}_t)$. Note that $\{(X_t,\mathcal{F}_t) : t \in T\}$ is a martingale. Remember that each X_t is defined only up to an almost sure equivalence. Question: Must X be progressively measurable?

To keep things simple, assume $T = [0,1]$.

Suppose Y_t is another choice for $\mathbb{P}(\xi \mid \mathcal{F}_t)$. Note that

$$N_t := \{\omega : X_t(\omega) \ne Y_t(\omega)\} \in \mathcal{N}$$

That is, the stochastic process Y is a ***version*** of X. However, the sample paths of X and Y can be very different:

$$\{\omega : X(\cdot,\omega) \ne Y(\cdot,\omega)\} \subseteq \cup_{t\in T} N_t$$

A union of uncountably many negligible sets need not be negligible.

We need to be careful about the choice of the random variable from the equivalence class corresponding to $\mathbb{P}(\xi \mid \mathcal{F}_t)$.

How to construct a cadlag version of X

Without loss of generality (why?) suppose $\xi \geq 0$.

- First build a nice "dense skeleton". Define $S_k := \{i/2^k : i = 0, 1, \dots, 2^k\}$ and $S = \cup_{k\in\mathbb{N}} S_k$. For each s in S, choose arbitrarily a random variable X_s from the equivalence class $\mathbb{P}(\xi \mid \mathcal{F}_s)$.
- Show that
$$\mathbb{P}\{\max_{s\in S_k} X_s > x\} \leq \mathbb{P}X_0/x \qquad \text{for each } x > 0.$$
Let k tend to infinity then x tend to infinity to deduce that $\sup_{s\in S} X_s < \infty$ almost surely.
- For fixed rational numbers $0 < \alpha < \beta$, invoke Dubin's inequality to show that the event
$$A(\alpha, \beta, k, n) := \{\text{the process } \{X_s : s \in S_k\} \text{ makes at least } n \text{ upcrossings of } [a, \beta]\ \}$$
has probability less than $(\alpha/\beta)^n$.
- Let k tend to infinity, then n tend to infinity, then take a union over rational pairs to deduce existence of an $N \in \mathcal{N}$ such that, for $\omega \in N^c$, the sample path $X(\cdot, \omega)$ (as a function on S) is bounded and
$$X(\cdot, \omega) \text{ makes only finitely many upcrossings of each rational interval .}$$
- Deduce that $\widetilde{X}_t(\omega) := \lim_{s\downarrow\downarrow t} X(s, \omega)$ exists and is finite for each $t \in [0.1)$ and each $\omega \in N^c$. Deduce also that $\lim_{s\uparrow\uparrow t} X(s, \omega)$ exists and is finite for each $t \in (0.1]$ and each $\omega \in N^c$.

$\uparrow\uparrow$ means strictly increasing and $\downarrow\downarrow$ means strictly decreasing

- Define $\widetilde{X}(\cdot, \omega) \equiv 0$ for $\omega \in N$. Show that $\widetilde{X}$ has cadlag sample paths.
- Note: $\widetilde{X}$ need not be $\mathcal{F}_t$-measurable but it is measurable with respect to the sigma-field $\widetilde{\mathcal{F}}_t := \cap_{s>t}\sigma(\mathcal{N} \cup \mathcal{F}_t)$.
- Show that $\{\widetilde{\mathcal{F}}_t : t \in [0, 1]\}$ is ***right continuous***, that is, $\widetilde{\mathcal{F}}_t = \cap_{s>t}\widetilde{\mathcal{F}}_s$, and that $\mathcal{N} \subseteq \widetilde{\mathcal{F}}_t$. [Assuming that $\mathbb{P}$ is complete, a filtration with these properties is said to be ***standard*** or to satisfy the ***usual conditions***.]
- Show that $\{(\widetilde{X}_t, \widetilde{\mathcal{F}}_t) : 0 \leq t \leq 1\}$ is a martingale with cadlag sample paths.
- Is it true that $\widetilde{X}$ is a version of X?

To complete your understanding, find a filtration (which is necessarily not standard) for which there is a martingale that does not have a version with cadlag sample paths.

Why do you think that most authors prefer to assume the usual conditions?

Small exercise on measurability

Suppose X is a bounded random variable and that $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$. Suppose that for each $\epsilon > 0$ there is a finite set of $\mathcal{G}$-questions (that is, you learn the value of $\{\omega \in G_i\}$ for some sets of your choosing $G_1, \dots, G_N$ from $\mathcal{G}$) from which $X(\omega)$ can be determined up to a $\pm\epsilon$ error. Show that X is $\mathcal{G}$-measurable. [This problem might help you think about measurability. Imagine that you are allowed to ask the value of $\{\omega \in F\}$, but I will answer only if F is a set from $\mathcal{G}$.]