PROJECT 10

1. Change of measure for Brownian motion

Let $\{B_t : 0 \le t \le 1\}$ be a Brownian motion with respect to a (standard) filtration $\{\mathcal{F}_t\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Write $\mathcal{U}$ for its quadratic variation process, $\mathcal{U}_t = t$.

For each $\alpha \in \mathbb{R}$, the process

$$q_t = \exp\left(\alpha B_t - \tfrac{1}{2}\alpha^2 t\right) \qquad \text{for } 0 \le t \le 1$$

is a nonnegative martingale, with $\mathbb{P}q_t = \mathbb{P}q_0 = 1$. Define a new probability measure $\mathbb{Q}_\alpha$ on $\mathcal{F}_1$ by specifying q_1 to be its density with respect to $\mathbb{P}$. That is,

$$\mathbb{Q}_\alpha X = \mathbb{P}(Xq_1)$$

at least for all bounded random variables X.

- Show that $\mathbb{Q}_\alpha$ is equivalent to $\mathbb{P}$, in the sense that both measures have the same collection $\mathcal{N}$ of negligible sets.
- Show that $\mathbb{Q}_\alpha X = \mathbb{P}(Xq_t)$ if X is $\mathcal{F}_t$-measurable. Explain why q_t is a Radon-Nikodym density for $\mathbb{Q}_\alpha$ with respect to $\mathbb{P}$ when both measures are restricted to $\mathcal{F}_t$.
- For fixed s and $t = s + \delta$, a fixed F in $\mathcal{F}_s$, and a bounded measurable f, show that

$$\begin{aligned}\mathbb{Q}_\alpha F f(B_t - B_s) &= \mathbb{P}(Fq_s)\,\mathbb{P}\left(f(B_t - B_s)\exp\left(\alpha(B_t - B_s) - \tfrac{1}{2}\alpha^2\delta\right)\right)\\ &= \mathbb{Q}_\alpha F \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\delta}} f(z)\exp\left(-\tfrac{1}{2}(z-\alpha\delta)^2/\delta\right)\,dz\end{aligned}$$

- Deduce that, under $\mathbb{Q}_\alpha$, the process $B_t - \alpha t$ is a standard Brownian motion.

2. The Black-Scholes formula

Stock prices (in units so that $S_0 \equiv 1$) are sometimes modeled by a continuous process driven by a Brownian motion, B, on $[0, 1]$;

$$\begin{aligned}S_t &= \exp((\mu - 1/2\sigma^2)t + \sigma B_t) \qquad \text{for } 0 \le t \le 1\\ &= \exp(\sigma\widetilde{B}_t - 1/2\sigma^2 t) \qquad \text{where } \widetilde{B}_t = B_t + (\mu/\sigma)t\end{aligned}$$

for constants $\sigma > 0$ (assumed known) and μ (unknown). That is,

$$S_t = \psi(B_t, \mathcal{U}_t) \qquad \text{where } \psi(x, y) = \exp(\sigma x + (\mu - 1/2\sigma^2)y).$$

I am ignoring inflation. cf. expression of value of stock as a multiple of a bond price.

> Suppose $Y = f(S)$, with f a $\mathcal{C}$-measurable functional on $C[0, 1]$. How much should one pay at time 0 in order to receive the amount Y at time 1?

- Use the Itô formula to show that

<1> $$S_t = 1 + \sigma S \bullet B_t + \mu S \bullet \mathcal{U}_t,$$

> In more traditional notation,
>
> $$dS_t = \sigma S_t\,dB_t + \mu S_t\,dt, \qquad \text{or} \qquad \frac{dS_t}{S_t} = \sigma\,dB_t + \mu\,dt.$$
>
> Roughly speaking, the relative increments of S behave like the increments of a Brownian motion with drift μ. The process $\sigma S \bullet B$ is the $\text{loc}\mathcal{M}_0^2[0,1]$ part of the semimartingale decomposition of S.

- Similarly, show that $S_t = 1 + \sigma S \bullet \widetilde{B}_t$.
- Show that Y can be written as a $\mathcal{C}$-measurable functional of the $\widetilde{B}$ sample path.
- Temporarily suppose that $\mu = 0$, so that $\widetilde{B}$ is a standard Brownian motion.
  (i) Use stochastic calculus to show that
  $$\widetilde{B} = \frac{1}{\sigma S} \bullet S$$
  Hint: What do you know about the increments of the process that takes a constant value?
  (ii) Suppose $\mathbb{P}Y^2 < \infty$. Invoke results from Project 9 to show that there exists a predictable H such that

<2> $$Y = \mathbb{P}Y + H \bullet \widetilde{B}_1 = \mathbb{P}Y + K \bullet S_1 \qquad \text{where } K := \frac{H}{\sigma S}.$$

  (iii) Interpret the last equality as an assertion that there exists an (idealized?) hedging stategy that returns $Y - \mathbb{P}Y$. Deduce that the arbitrage price for Y equals $\mathbb{P}Y$ in the special case where $\mu = 0$.
- Now consider the case where μ is unknown, possibly nonzero. Let $\mathbb{Q}_\alpha$ be the probability measure with density $\exp(\alpha B_1 - \frac{1}{2}\alpha^2)$ with respect to $\mathbb{P}$, where $\alpha = -\mu/\sigma$. Show that $\widetilde{B}$ is a standard Brownian motion under $\mathbb{Q}_\alpha$.
- Assume that $\mathbb{Q}_\alpha Y^2 < \infty$. Show that there exists some predictable process K_α (in some apppropriate $\mathcal{L}^2$ space) for which
  $$Y = \mathbb{Q}_\alpha Y + K_\alpha \bullet S_1 \qquad \text{almost surely } [\mathbb{Q}_\alpha].$$
- I believe that the threat of the trading scheme that delivers a return $K_\alpha \bullet S_1$ now forces $\mathbb{Q}_\alpha Y$ to be the amount one should pay at time 0 to receive the amount Y at time 1. What do you think? Should the fact that K_α seems to depend on the unknown μ invalidate the arbitrage argument?
- Suppose Y actually depends only on the stock price at time 1, that is, $Y = f_1(S_1)$ for some measurable function f_1. Show that
  $$\mathbb{Q}_\alpha Y = \mathbb{Q} f_1\left(\exp(\sigma W - \tfrac{1}{2}\sigma^2)\right) \qquad \text{where } W \sim N(0,1) \text{ under } \mathbb{Q}.$$
  Deduce that $\mathbb{Q}_\alpha Y$ does not depend on μ.
- Specialize even further, to the case where $f_1(x) = (x - C)^+$, for some constant C, to derive the famous Black-Scholes formula for the price of a European option.

3. Does K_α actually depend on μ?

As I type this Project late at night, I find myself in the embarrassing position of not really understanding how the question is handled for a general Y. However, when $Y = f_1(S_1)$ there is another approach that avoids the difficulty by constructing an explicit strategy via the solution to a partial differental equation.

I should reread Harrison & Pliska (1981).

Look for a smooth function $f(x,t)$ for which $f(x,1) = f_1(x)$ and
$$\sigma^2 x^2 f_{xx}(x,t) + f_t(x,t) = 0$$

- Use Itô to show that
  $$f(S_t, t) = f(1,0) + F_x \bullet S_t \qquad \text{almost surely } [\mathbb{P}].$$
- Show that, under $\mathbb{Q}_\alpha$, the stock price process is a martingale. Deduce that
  $$f(S_t, t) = \mathbb{Q}_\alpha\left(f(S_1, 1) \mid \mathcal{F}_t\right) = \mathbb{Q}_\alpha(Y \mid \mathcal{F}_t)$$

and, in particular, $f(1,0) = \mathbb{Q}_\alpha(Y \mid \mathcal{F}_0) = \mathbb{Q}_\alpha Y$.

I hope I can sort through my confusion before the lecture. I will reread the final section of Chung & Williams (1990).

4. Change of measure for semimartingales

The key fact about the change from $\mathbb{P}$ to an equivalent measure $\mathbb{Q}$ is the preservation of the semimartingale property. It is not at all an obvious fact. For suppose that X is a $\mathbb{P}$-semimartingale that has decomposition $X_0 + M + A$, where M is a locally square integrable $\mathbb{P}$-martingale and $A \in \mathcal{FV}_0$. Under $\mathbb{Q}$ the A process is still in $\mathcal{FV}_0$, but we will have to subtract another $\mathcal{FV}_0$ process A^* from M to make it a locally square integrable $\mathbb{Q}$-martingale, leading to the $\mathbb{Q}$-semimartingale decomposition $X = X_0 + (M - A^*) + (A + A^*)$.

To establish these facts in the general case I would need some theory about processes with jumps—things like the Doob-Meyer decomposition. Using only tools developed in the course, I can show you how to treat a special case.

Consider only a process $M \in \text{loc}\mathcal{M}_0^2([0,1],\mathbb{P})$ with continuous sample paths. Here I have added the $\mathbb{P}$ to emphasize that the martingale properties hold under the $\mathbb{P}$ distribution. Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, with $q_1 := d\mathbb{Q}/d\mathbb{P}$ and $d\mathbb{P}/d\mathbb{Q} = p_1 = 1/q_1$. Assume that the cadlag versions of the $\mathbb{P}$-martingale $q_t := \mathbb{P}(q_1 \mid \mathcal{F}_t)$ and the $\mathbb{Q}$-martingale $p_t := \mathbb{Q}(p_1 \mid \mathcal{F}_t)$ actually have continuous sample paths.

- Show that $p_\sigma = \mathbb{Q}(p_1 \mid \mathcal{F}_\sigma)$ for each $[0,1]$-valued stopping time σ.
- Explain why we can assume $p_t q_t \equiv 1$. More specifically, explain why p_t can be thought of as the density of $\mathbb{P}$ with respect to $\mathbb{Q}$ when both measures are restricted to $\mathcal{F}_t$.

Define

$$\tau_k := 1 \wedge \inf\{t : p_t \geq k \text{ or } p_t \leq 1/k\} \wedge \inf\{t : |M_t| \geq k\}.$$

Without loss of generality, we may also assume that $M_{\wedge\tau_k} \in \mathcal{M}_0^2([0,1],\mathbb{P})$.

- Show that $pM \in \text{loc}\mathcal{M}_0^2([0,1],\mathbb{Q})$. Hint: For $s < t$ and $F \in \mathcal{F}_s$ show that
$$\mathbb{Q}F\left(p_{t\wedge\tau_k}M_{t\wedge\tau_k} - p_{s\wedge\tau_k}M_{s\wedge\tau_k}\right) = \mathbb{Q}F\left(p_{t\wedge\tau_k}M_{t\wedge\tau_k} - p_{s\wedge\tau_k}M_{s\wedge\tau_k}\right)\{\tau_k > s\}$$
Argue that $F\{\tau_k > s\} \in \mathcal{F}_{s\wedge\tau_k}$ then deduce that the right-hand side of the last equality equals $\mathbb{P}F\left(M_{t\wedge\tau_k} - M_{s\wedge\tau_k}\right)\{\tau_k > s\} = 0$.
- Use the fact that q and M are both in $\text{loc}\mathcal{M}_0^2([0,1],\mathbb{P})$ to explain why the process $Y := qM - V$, where $V := [q, M] \in \mathcal{FV}_0$, is also in $\text{loc}\mathcal{M}_0^2([0,1],\mathbb{P})$. Hint: First explain why $Y_{t\wedge\tau_k} = q \bullet M_{t\wedge\tau_k} + M \bullet q_{t\wedge\tau_k}$.
- Explain why both Y and V have continuous sample paths.
- Explain why $pY \in \text{loc}\mathcal{M}_0^2([0,1],\mathbb{Q})$.
- Explain why $[p, V] \equiv 0$. Hint: V is a $\mathcal{FV}_0$ process with continuous sample paths.
- Deduce that $p_t V_t = p \bullet V_t + V \bullet p_t$.
- Deduce that $M - p \bullet V - V \bullet p \in \text{loc}\mathcal{M}_0^2([0,1],\mathbb{Q})$.

- Explain why $V \bullet p \in \text{loc}\mathcal{M}_0^2([0,1],\mathbb{Q})$. Hint: p is a $\mathbb{Q}$-martingale.
- Explain why $A := p \bullet V$ is in $\mathcal{FV}_0$.
- Conclude that $M - A \in \text{loc}\mathcal{M}_0^2([0,1],\mathbb{Q})$.

You should check that this recipe works for the Brownian motion example in Section 1.

5. Things I could show if I had more time

(Actually I would also need some facts about processes with jumps.)

(i) Every local-martingale is a semimartingale.

(ii) Suppose $\mathbb{P}$ and $\mathbb{Q}$ are equivalent probability measures. If X is a $\mathbb{P}$-semimartingale then it is also a $\mathbb{Q}$-semimartingale. Moreover, for $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$, the stochastic integral $H \bullet X$ when calculated using the methods from Project 7 under $\mathbb{P}$ is the same as the stochastic integral when calculated under $\mathbb{Q}$. The last assertion can be proved using the characterization of the stochastic integral given in Project 7.

References

Chung, K. L. & Williams, R. J. (1990), *Introduction to Stochastic Integration*, Birkhäuser, Boston.

Harrison, J. M. & Pliska, S. R. (1981), 'Martingales and stochastic integrals in the theory of continuous trading', *Stochastic Processes and their Applications* **11**, 215–260.