Project 11

Notation: Write $\mathbb{P}_t(\ldots)$ for $\mathbb{P}(\ldots \mid \mathcal{F}_t)$ and $\text{var}_t(\ldots)$ for the corresponding conditional variance.

1. Diffusion heuristics

The rough idea of an Itô diffusion is: $\{X_t : t \in \mathbb{R}^+\}$ is adapted with continuous sample paths; and for small $\delta > 0$, with $\Delta X = X_{t+\delta} - X_t$,

<1> $$\mathbb{P}_t\left(\Delta X\right) \approx \delta b(X_t)$$

<2> $$\text{var}_t\left(\Delta X\right) \approx \delta \sigma^2(X_t)$$

where $b(\cdot)$ and $\sigma(\cdot)$ are deterministic functions. In what follows, both b and σ will be continuous functions.

Interpret <1> to mean that

$$\mathbb{P}_t(\Delta Z) \approx 0 \qquad \text{where } Z_t = X_t - \int_0^t b(X_s)\,ds.$$

More precisely, interpret <1> to mean that Z is a martingale with continuous sample paths and $Z_0 = 0$. Similarly, interpret <2> to mean $\mathbb{P}_t(\Delta Z)^2 \approx \delta\sigma^2(X_t)$, or

$$W_t := [Z, Z]_t - \int_0^t \sigma^2(X_s)\,ds \qquad \text{is a martingale.}$$

Note that W has continuous paths of finite variation. From the Problems to Project 9, we must have $W_t \equiv W_0 = 0$. That is, $[Z, Z]_t = \int_0^t \sigma^2(X_s)\,ds$.

Put another way, we could interpret <1> and <2> to mean that

<3> $$X_t = x_0 + Z_t + b(X) \bullet \mathcal{U}_t \qquad \text{where } X_0 = x_0$$

with Z a (local?) martingale for which $[Z, Z] = \sigma^2(X) \bullet \mathcal{U}$. Here, and subsequently, I am abusing notation by writing $b(X)$ for the process that takes the value $b(X_s)$ at time s, and so on.

Note that $\sigma^2(X)$ is adapted and has continuous paths

Suppose there exist processes X and Z with the properties just described. If $\sigma(x) \neq 0$ for all x then $1/\sigma(X)$ is locally bounded and predictable. The process $B := (1/\sigma(X)) \bullet Z$ is a local martingale, with continuous sample paths, $B_0 = 0$, and

$$[B, B] = (1/\sigma^2(X)) \bullet [Z, Z] = \mathcal{U}.$$

Compare with the argument in Stroock & Varadhan (1979, Section 4.5)

That is, by the Lévy characterization, B is a Brownian motion for which

<4> $$X_t = x_0 + \sigma(X) \bullet B_t + b(X) \bullet \mathcal{U}_t$$

Many authors would write the last representation as

<5> $$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt$$

and call it a ***stochastic differential equation*** for X with initial condition $X_0 = x_0$.

If the representation <3> were valid, and if f were twice continuously differentiable, Itô's formula would give

$$\begin{aligned} f(X_t) &= f(x_0) + f'(X) \bullet (Z + b(X) \bullet \mathcal{U})_t + \tfrac{1}{2} f''(X) \bullet [Z, Z]_t \\ &= f(x_0) + f'(X) \bullet Z_t + \left(\tfrac{1}{2}\sigma^2(X) f''(X) + b(X) f'(X)\right) \bullet \mathcal{U}_t \end{aligned}$$

This representation would imply that

<6> $$f(X_t) - \left(\tfrac{1}{2}\sigma(X)^2 f''(X) + b(X) f'(X)\right) \bullet \mathcal{U}_t \qquad \text{is a martingale}$$

for each suitably smooth f.

The question of whether an X satisfying <4> or <6>actually exists, and to what extent it is uniquely determined, is the subject of a huge literature. The small sampling that follows is based mostly on

(i) Stroock & Varadhan (1979, Chapters 4 and 5),
(ii) Durrett (1984, Chapter 9)
(iii) Chung & Williams (1990, Chapter 10).

2. Existence and uniqueness of a solution to a SDE

SDE = stochastic differential equation

Seek a solution for the SDE <5> with initial condition $X_0 \equiv x_0$, for a fixed $x_0 \in \mathbb{R}$. Suppose the functions b and σ satisfy the following conditions for some finite constant C:

<7>
$$\begin{cases} |b(x)| \le C, & |\sigma(x)| \le C \qquad \text{for all } x \\ |b(x) - b(y)| \le C|x-y|, & |\sigma(x) - \sigma(y)| \le C|x-y| \qquad \text{for all } x \text{ and } y \end{cases}$$

I am so very lazy to use the same constant for all the bounds.

Assume a standard Brownian motion B is given. Start by building the solution on a fixed interval $[0, T]$. Define $X^{(0)} \equiv x_0$ and, for $n \ge 0$,

$$X_t^{(n+1)} = x_0 + \sigma(X^{(n)}) \bullet B_t + b(X^{(n)}) \bullet \mathfrak{U}_t$$

Define

$$\Delta_{n+1}(t) := \mathbb{P}\sup_{s \le t} |X_s^{(n+1)} - X_s^{(n)}|^2.$$

- Show that $\Delta_1(T) \le c_0 := 8C^2T + 2C^2T^2$, or something like that.
- For $n \ge 1$ show that

$$\begin{aligned} \Delta_{n+1}(T) &\le 2\mathbb{P}\sup_{t \le T} |\sigma(X^{(n)}) \bullet B_t - \sigma(X^{(n-1)}) \bullet B_t|^2 \\ &\quad + 2\mathbb{P}\sup_{t \le T} \left|\int_0^t b(X_s^{(n)}) - b(X^{(n-1)})\, ds\right|^2 \\ &\le 8\mathbb{P}|\sigma(X^{(n)}) \bullet B_T - \sigma(X^{(n-1)}) \bullet B_T|^2 \\ &\quad + 2T^2\mathbb{P}\left(\frac{1}{T}\int_0^T |b(X_s^{(n)}) - b(X_s^{(n-1)})|\, ds\right)^2 \\ &\le 8\int_0^T \mathbb{P}|\sigma(X_s^{(n)}) - \sigma(X_s^{(n-1)})|^2 \\ &\quad + 2T^2\mathbb{P}\left(\frac{1}{T}\int_0^T |b(X_s^{(n)}) - b(X_s^{(n-1)})|\, ds\right)^2 \\ &\le K_T \int_0^T \Delta_n(s)\, ds, \end{aligned}$$

  where K_T is a constant that depends on T.
- Strengthen the previous result to

$$\Delta_{n+1}(t) \le K_T \int_0^t \Delta_n(s)\, ds \qquad \text{for all } t \in [0, T].$$

- Show that

$$\begin{aligned} \Delta_{n+1}(T) &\le K_T^n \int \dots \int \{0 \le t_1 \le t_2 \le \dots \le t_n \le T\} \Delta_1(t_1)\, dt_1 dt_2 \dots dt_n \\ &\le c_0 (TK_T)^n / n! \end{aligned}$$

- Deduce that

$$\mathbb{P}\sum\nolimits_{n \ge 1} \sup_{s \le T} |X_s^{(n+1)} - X_s^{(n)}| < \infty$$

- Deduce that there exists an adapted process $\{X_t : 0 \le t \le T\}$ with continuous sample paths, such that
$$\sup_{s\le T} |X_s^{(n)} - X_s| \to 0 \qquad \text{almost surely.}$$
- Deduce that
$$\sup_{s\le T} \left(|b(X_s^{(n)}) - b(X_s)| + |\sigma(X_s^{(n)}) - \sigma(X_s)|\right) \to 0 \qquad \text{almost surely.}$$
- Deduce that
$$|\sigma(X^{(n)}) \bullet B - \sigma(X) \bullet B| + |b(X^{(n)}) \bullet \mathcal{U} - b(X) \bullet \mathcal{U}| \overset{ucpc}{\longrightarrow} 0$$
- Conclude that $\{X_t : 0 \le t \le T\}$ satisfies the SDE <5> with initial condition $X_0 \equiv x_0$.
- Suppose $\{Y_t : 0 \le t \le T\}$ is another solution to the SDE with the same initial condition. Define
$$\Delta(t) := \mathbb{P} \sup_{s\le t} |X_s - Y_s|^2.$$
Show that for some constants c_1 and κ, which might depend on T,
$$\Delta(T) \le \left(c_1 \kappa^n / n!\right) \Delta(T).$$
Deduce that $\Delta(T) = 0$ and hence
$$\mathbb{P}\{\omega : \exists t \le T \text{ with } X_t(\omega) \ne Y_t(\omega)\} = 0.$$
- Suppose $\{X_t : 0 \le t \le T_1\}$ and $\{Z_t : 0 \le t \le T_2\}$ are solutions to the SDE over different ranges, $[0, T_1]$ and $[0, T_2]$, with $X_0 = Z_0 = x_0$. Show that almost all paths $X(\cdot, \omega)$ and $Z(\cdot, \omega)$ agree on the interval $[0, T_1 \wedge T_2]$. Explain how this result enables us to find a unique solution (up to almost sure equivalence) on $\mathbb{R}^+$.

3. Dependence of the solution on B: strong and weak solutions of the SDE

The solution X constructed in Section 2 depends only on the Brownian motion. More precisely, we could choose $\{\mathcal{F}_t\}$ as the augmented Brownian filtration and have X adapted to that filtration.

- Try to make some sense of the last assertion. Perhaps you could argue inductively that each approximation $X^{(n)}$ is adapted to the augmented filtration. I would like to show that this means we can choose $X_t(\omega)$ as $f(B_{\wedge t}(\omega), t)$ for some suitably measurable function $f : C(\mathbb{R}^+) \times \mathbb{R}^+ \to \mathbb{R}$. Perhaps we could require $t \mapsto f(y, t)$ to be continuous for each fixed y.

The idea is that B can provide both the filtration and the process for the stochastic integral $\sigma(X) \bullet B$. I think this is what it means for X to be a ***strong solution*** of SDE. Clearly, if we start from a different Brownian motion then we get a different solution.

The distribution of X is a probability measure, $\mathbb{Q}_{x_0}$, on the cylinder sigma-field $\mathcal{C}$ of $C(\mathbb{R}^+)$. More formally, if we can regard f as a $\mathcal{C}\backslash\mathcal{C}$-measurable map from $C(\mathbb{R}^+)$ back into itself, then $\mathbb{Q}_{x_0}$ is the image of Wiener measure $\mathbb{W}$ under the map f.

I am a lttle unsure of these assertions, because I have not worked through the whole construction myself. I am relying on what I think Durret and Chung&Williams are asserting.

I think that for some SDE's it is possible to prove the existence of a $\mathbb{Q}_{x_0}$ on $\mathcal{C}$ under which the coordinate map defines a process with continuous paths started at x_0 for which the analog of property <6> holds. Slight refinements of the arguments in Section 1 then show how to construct a Brownian motion B for which <4> holds.

For a famous example where there exists a (nonunique) weak solution but no strong solution see Chung & Williams (1990, Secton 10.4).

4. Relaxation of assumptions on b and σ

Localization arguments allow us to relax the conditions <7> on the functions $b(\cdot)$ and $\sigma(\cdot)$ to existence of constants C_r for each $R > 0$ such that

<8> $$\max\big(|b(x) - b(y)|, |\sigma(x) - \sigma(y)|\big) \le C_R|x - y| \qquad \text{if } \max(|x|, |y|) \le R.$$

Most authors seem also to require a growth condition,

$$\max\big(|b(x)|, |\sigma(x)|\big) = O(|x|) \qquad \text{as } |x| \to \infty.$$

Frankly, I do not really understand why the growth condition is needed.

It seems to me that assumption <8> implies existence of finite constants K_R for which

$$|b(x)| + |\sigma(x)| \le K_R \qquad \text{when } |x| \le R.$$

Define

$$b_R(x) := \max(-K_R, \min(b(x), K_R))$$
$$\sigma_R(x) := \max(-K_R, \min(\sigma(x), K_R))$$

An analog of <7> holds for b_R and σ_R. There exists continuous adapted processes for which

$$X_t^{(R)} = x_0 + \sigma_R(X^{(R)}) \bullet B_t + b_R(X^{(R)}) \bullet \mathcal{U}_t$$

Define $\tau_R := \inf\{t : |X_t^{(R)}| \ge R\}$. I think that

$$X_{t\wedge\tau_R}^{(R)} = x_0 + \sigma(X^{(R)}) \bullet B_{t\wedge\tau_R} + b(X^{(R)}) \bullet \mathcal{U}_{t\wedge\tau_R}$$

It should be possible to paste together the solutions $X^{(R)}$ for an increasing sequence of R values, invoking the uniqueness theorem from Section 2 to show that $X^{(2R)}$ agrees with $X^{(R)}$ at least until $|X^{(2R)}| \ge R$. If the corresponding stopping times τ_R were to increase to infinity as $R \uparrow \infty$ then we would get a solution to the original SDE. I think this is where the growth condition is needed.

I need to read the last part of Chung & Williams (1990, Secton 10.2) more carefully.

5. Examples

We should try to establish existence and uniqueness of the solutions to two simple SDE's:

(i) (geometric Brownian motion) Using the Itô formula, you showed in Project 10 that

$$X_t = \exp\big(\sigma B_t + (\mu - \tfrac{1}{2}\sigma^2)t\big)$$

is a solution to the equation $X_t = 1 + \sigma X \bullet B_t + \mu X \bullet \mathcal{U}_t$. Is it the only solution?

(ii) (Ornstein-Uhlenbeck process) By the Itô formula, the process

$$X_t = e^{-\alpha t}\big(x_0 + E \bullet B_t\big) \qquad \text{where } E_s := e^{\alpha s}$$

is a solution to the SDE $dX_t = -\alpha X_t + dB_t$ with $X_0 = x_0$, that is,

$$X_t = x_0 + B_t - \alpha X \bullet \mathcal{U}_t$$

Again, is it the only solution? Could we establish both existence and uniqueness of a (strong) solution by appeal to the general theory?

References

Chung, K. L. & Williams, R. J. (1990), *Introduction to Stochastic Integration*, Birkhäuser, Boston.

Durrett, R. (1984), *Brownian Motion and Martingales in Analysis*, Wadsworth, Belmont CA.

Stroock, D. W. & Varadhan, S. R. S. (1979), *Multidimensional Diffusion Processes*, Springer, New York.