Things to explain in your notebook:

(i) how to prove that a first passage time is a stopping time

(ii) how the Stopping Time Lemma extends from discrete to continuous time

(iii) Give some illuminating examples.

Fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Negligible sets \(N := \{N \in \mathcal{F} : \mathbb{P}N = 0\}\). Assume the probability space is complete, that is, for all \(A \subseteq \Omega\), if \(A \subseteq N \in N\) then \(A \in N\). From now on, unless indicated otherwise, also assume that all filtrations \(\{\mathcal{F}_t : t \in T\}\) are standard, that is, \(N \subseteq \mathcal{F}_t = \mathcal{F}_{t+}\) for each \(t\).

First passage times (a.k.a. debuts)

Suppose \(\{X_t : t \in \mathbb{R}^+\}\) is adapted and that \(B \in \mathcal{B}(\mathbb{R})\). Define

\[
\tau(\omega) = \inf\{t \in \mathbb{R}^+: X(t, \omega) \in B\}.
\]

As usual, \(\inf \emptyset := +\infty\).

- **Easy case**: \(B\) open and \(X\) has right-continuous paths
  
  Let \(S\) be a countable, dense subset of \(\mathbb{R}^+\). Show that
  \[
  \{\omega : \tau(\omega) < t\} = \bigcup_{s \in S, s < t} \{X_s(\omega) \in B\} \in \mathcal{F}_t.
  \]
  
  Deduce that \(\{\tau \leq t\} \in \mathcal{F}_{t+} = \mathcal{F}_t\).

- **Slightly harder case**: \(B\) closed and \(X\) has continuous paths
  
  Let \(G_i := \{x : d(x, B) < i^{-1}\}\), an open set. Define \(\tau_i = \inf\{t : X_t \in G_i\}\).
  Show that \(\tau = \sup_i \tau_i\) so that \(\{\tau \leq t\} = \bigcap_i \{\tau_i \leq t\} \in \mathcal{F}_t\).

- **General case**: \(B\) any Borel set and \(X\) progressively measurable
  
  See the handout on analytic sets. The idea is that the set
  
  \[
  D_t := \{(s, \omega) : s < t \text{ and } X(s, \omega) \in B\}
  \]

  is \(\mathcal{B} \otimes \mathcal{F}_t\)-measurable. The set \(\{\tau < t\}\) is the projection of \(D_t\) onto \(\Omega\). A deep result about analytic sets asserts that the projection of \(D_t\) belongs to the \(\mathbb{P}\)-completion of \(\mathcal{F}_t\). For a standard filtration, \(\mathcal{F}_t\) is already complete. Thus \(\{\tau < t\} \in \mathcal{F}_t\) and \(\{\tau \leq t\} \in \mathcal{F}_{t+} = \mathcal{F}_t\). That is, \(\tau\) is a stopping time.

Preservation of martingale properties at stopping times

**Stopping Time Lemma.** Suppose \(\{(X_t, \mathcal{F}_t) : 0 \leq t \leq 1\}\) is a positive supermartingale with cadlag sample paths. Suppose \(\sigma\) and \(\tau\) are stopping times taking values in \([0, 1]\) and \(F\) is an event in \(\mathcal{F}_\sigma\) for which \(\mathbb{P}\sigma(\omega) \leq \tau(\omega)\) when \(\omega \in F\). Then \(\mathbb{P}X_\sigma F \geq \mathbb{P}X_\tau F\).

**Proof:** discrete case. Suppose both stopping times actually take values in a finite subset of points \(t_0 < t_1 < \ldots < t_N\) in \([0, 1]\). Define \(\xi_i := X(t_i) - X(t_{i-1})\). The superMG property means that

\[
\mathbb{P}\xi_i F \leq 0 \quad \text{for all } F \in \mathcal{F}(t_{i-1})
\]

Note that

\[
X_\tau = X(t_0) + \sum_{i=1}^N \xi_i[t_i \leq \tau]
\]

\[
X_\sigma = X(t_0) + \sum_{i=1}^N \xi_i[t_i \leq \sigma]
\]

so that

\[
\mathbb{P}(X_\tau - X_\sigma) F = \sum_{i=1}^N \mathbb{P}(\xi_i[\sigma < t_i \leq \tau] F)
\]
The last equality uses the fact that \( \sigma \leq \tau \) on \( F \). Check that \( \{ \sigma < t \leq \tau \} F \) is \( \mathcal{F}(t_{-1}) \)-measurable.

**Proof: general case.** For each \( n \in \mathbb{N} \) define \( \sigma_n = 2^{-n}[2^n \sigma] \). That is, \[
\sigma_n(\omega) = 0[\sigma(\omega) = 0] + \sum_{i=1}^{2^n} i/2^n \{ (i-1)/2^n < \sigma(\omega) \leq i/2^n \}
\]

- Check that \( \sigma_n \) is a stopping time taking values in a finite subset of \([0, 1]\).
- Question: If we rounded down instead of up, would we still get a stopping time? Check that \( F \in \mathcal{F}(\sigma_n) \):
\[
F[\sigma_n \leq i/2^n] = F[\sigma \leq i/2^n] \in \mathcal{F}(i/2^n).
\]

Define \( \tau_n \) analogously.

- From the discrete case, deduce that
\[
\mathbb{P} X(\sigma_n) F \geq \mathbb{P} X(\tau_n) F \quad \text{for each } n
\]

- Show that \( \sigma_n(\omega) \downarrow \sigma(\omega) \) and \( \tau_n(\omega) \downarrow \tau(\omega) \) as \( n \to \infty \).
- Use right-continuity of the sample paths to deduce that \( X(\sigma_n, \omega) \to X(\sigma, \omega) \) and \( X(\tau_n, \omega) \to X(\tau, \omega) \) for each \( \omega \).
- Prove that \( \{X(\sigma_n) : n \in \mathbb{N}\} \) is uniformly integrable. Write \( Z_n \) for \( X(\sigma_n) \).
  
  (i) First show that \( \mathbb{P} Z_n \uparrow c_0 \leq \mathbb{P} X_0 \) as \( n \to \infty \).
  
  (ii) Choose \( m \) so that \( \mathbb{P} Z_m > c_0 - \epsilon \). For \( n \geq m \), show that \( Z_n, Z_{n-1}, \ldots, Z_m \) is a superMG.

  (iii) For constant \( K \) and \( n \geq m \), show that
\[
\mathbb{P} Z_n[Z_n \geq K] = \mathbb{P} Z_n - \mathbb{P} Z_n[Z_n < K] \\
\leq c_0 - \mathbb{P} Z_m[Z_m < K] \\
\leq \epsilon + \mathbb{P} Z_m[Z_m \geq K]
\]

  (iv) Show that \( \mathbb{P} \{Z_n \geq K\} \leq c_0/K \), then complete the proof of uniform integrability.

- Prove similarly that \( \{X(\tau_n) : n \in \mathbb{N}\} \) is uniformly integrable. Pass to the limit in the “discretized version” to complete the proof.

**Problems = some possible examples for your notes**

[1] Show that Lemma <1> also holds without the assumption that \( X_t \geq 0 \). Hint: Let \( M \) be a cadlag version of the martingale \( \mathbb{P} (X^- | \mathcal{F}_t) \). Show that \( Z_t := X_t + M_t \) is a positive superMG with cadlag paths.

[2] Suppose \( \{(X_t, \mathcal{F}_t) : t \in \mathbb{R}^+\} \) is a positive supermartingale with cadlag sample paths. Suppose \( \sigma_1 \leq \sigma_2 \leq \ldots \) are stopping times taking values in \([0, \infty)\).

  (i) Show that the sequence \( X_{\sigma_n}[\sigma_n < \infty] \) is a superMG for a suitable filtration. Compare with UGMTPT Problem 6.5.

  (ii) If \( X \) is actually a positive martingale, is the sequence \( X_{\sigma_n}[\sigma_n < \infty] \) also a martingale?

  (iii) Same question as for part (ii), except that the \( \sigma_n \) all take values in \([0, 1]\).

[3] Suppose \( \{(X_t, \mathcal{F}_t) : t \in [0, 1]\} \) is a positive supermartingale with cadlag sample paths. Let \( \mathcal{S} \) denote the set of all stopping times taking values in \([0, 1]\). Is the set of random variables \( \{X_{\sigma} : \sigma \in \mathcal{S}\} \) uniformly integrable? It might help to track down the concept of superMGs of class [D].