Project 2

Things to explain in your notebook:

(i) how to prove that a first passage time is a stopping time

(ii) how the Stopping Time Lemma extends from discrete to continuous time

(iii) Give some illuminating examples.

Fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Negligible sets $\mathcal{N} := \{N \in \mathcal{F} : \mathbb{P}N = 0\}$. Assume the probability space is complete, that is, for all $A \subseteq \Omega$, if $A \subseteq N \in \mathcal{N}$ then $A \in \mathcal{N}$. From now on, unless indicated otherwise, also assume that all filtrations $\{\mathcal{F}_t : t \in T\}$ are standard, that is, $\mathcal{N} \subseteq \mathcal{F}_t = \mathcal{F}_{t+}$ for each t.

First passage times (a.k.a. debuts)

Suppose $\{X_t : t \in \mathbb{R}^+\}$ is adapted and that $B \in \mathcal{B}(\mathbb{R})$. Define

$$\tau(\omega) = \inf\{t \in \mathbb{R}^+ : X(t, \omega) \in B\}.$$

As usual, $\inf \emptyset := +\infty$.

- *Easy case: B open and X has right-continuous paths*
  Let S be a countable, dense subset of $\mathbb{R}^+$. Show that
  $$\{\omega : \tau(\omega) < t\} = \cup_{t>s\in S}\{X_s(\omega) \in B\} \in \mathcal{F}_t$$
  Deduce that $\{\tau \le t\} \in \mathcal{F}_{t+} = \mathcal{F}_t$.
- *Slightly harder case: B closed and X has continuous paths*
  Let $G_i := \{x : d(x, B) < i^{-1}\}$, an open set. Define $\tau_i = \inf\{t : X_t \in G_i\}$. Show that $\tau = \sup_i \tau_i$ so that $\{\tau \le t\} = \cap_{i\in\mathbb{N}}\{\tau_i \le t\} \in \mathcal{F}_t$.
- *General case: B any Borel set and X progressively measurable*
  See the handout on analytic sets. The idea is that the set
  $$D_t := \{(s, \omega) : s < t \text{ and } X(s, \omega) \in B\}$$
  is $\mathcal{B}_t \otimes \mathcal{F}_t$-measurable. The set $\{\tau < t\}$ is the projection of D_t onto Ω. A deep result about analytic sets asserts that the projection of D_t belongs to the $\mathbb{P}$-completion of $\mathcal{F}_t$. For a standard filtration, $\mathcal{F}_t$ is already complete. Thus $\{\tau < t\} \in \mathcal{F}_t$ and $\{\tau \le t\} \in \mathcal{F}_{t+} = \mathcal{F}_t$. That is, τ is a stopping time.

Preservation of martingale properties at stopping times

<1> **Stopping Time Lemma.** *Suppose $\{(X_t, \mathcal{F}_t) : 0 \le t \le 1\}$ is a positive supermartingale with cadlag sample paths. Suppose σ and τ are stopping times taking values in $[0, 1]$ and F is an event in $\mathcal{F}_\sigma$ for which $\sigma(\omega) \le \tau(\omega)$ when $\omega \in F$. Then $\mathbb{P}X_\sigma F \ge \mathbb{P}X_\tau F$.*

bounded stopping times

Proof: discrete case. Suppose both stopping times actually take values in a finite subset of points $t_0 < t_1 < \ldots < t_N$ in $[0, 1]$. Define $\xi_i := X(t_i) - X(t_{i-1})$. The superMG property means that

$$\mathbb{P}\xi_i F \le 0 \qquad \text{for all } F \in \mathcal{F}(t_{i-1})$$

Note that

$$X_\tau = X(t_0) + \sum\nolimits_{i=1}^N \xi_i\{t_i \le \tau\}$$
$$X_\sigma = X(t_0) + \sum\nolimits_{i=1}^N \xi_i\{t_i \le \sigma\}$$

so that

$$\mathbb{P}(X_\tau - X_\sigma)F = \sum\nolimits_{i=1}^N \mathbb{P}\left(\xi_i\{\sigma < t_i \le \tau\}F\right)$$

The last equality uses the fact that $\sigma \le \tau$ on F. Check that $\{\sigma < t_i \le \tau\}F$ is $\mathcal{F}(t_{i-1})$-measurable.

Proof: general case. For each $n \in \mathbb{N}$ define $\sigma_n = 2^{-n}\lceil 2^n\sigma\rceil$. That is,

$$\sigma_n(\omega) = 0\{\sigma(\omega) = 0\} + \sum_{i=1}^{2^n} i/2^n\{(i-1)/2^n < \sigma(\omega) \le i/2^n\}$$

- Check that σ_n is a stopping time taking values in a finite subset of $[0, 1]$. Question: If we rounded down instead of up, would we still get a stopping time? Check that $F \in \mathcal{F}(\sigma_n)$:

  $$F\{\sigma_n \le i/2^n\} = F\{\sigma \le i/2^n\} \in \mathcal{F}(i/2^n).$$

  Define τ_n analogously.
- From the discrete case, deduce that

  $$\mathbb{P}X(\sigma_n)F \ge \mathbb{P}X(\tau_n)F \qquad \text{for each } n$$

- Show that $\sigma_n(\omega) \downarrow \sigma(\omega)$ and $\tau_n(\omega) \downarrow \tau(\omega)$ as $n \to \infty$.

left-continuous paths wouldn't help—why not?

- Use right-continuity of the sample paths to deduce that $X(\sigma_n, \omega) \to X(\sigma, \omega)$ and $X(\tau_n, \omega) \to X(\tau, \omega)$ for each ω.
- Prove that $\{X(\sigma_n) : n \in \mathbb{N}\}$ is uniformly integrable. Write Z_n for $X(\sigma_n)$.
  (i) First show that $\mathbb{P}Z_n \uparrow c_0 \le \mathbb{P}X_0$ as $n \to \infty$.
  (ii) Choose m so that $\mathbb{P}Z_m > c_0 - \epsilon$. For $n \ge m$, show that $Z_n, Z_{n-1}, \dots, Z_m$ is a superMG.
  (iii) For constant K and $n \ge m$, show that

  $$\begin{aligned}\mathbb{P}Z_n\{Z_n \ge K\} &= \mathbb{P}Z_n - \mathbb{P}Z_n\{Z_n < K\}\\ &\le c_0 - \mathbb{P}Z_m\{Z_n < K\}\\ &\le \epsilon + \mathbb{P}Z_m\{Z_n \ge K\}\end{aligned}$$

  (iv) Show that $\mathbb{P}\{Z_n \ge K\} \le c_0/K$, then complete the proof of uniform integrability.
- Prove similarly that $\{X(\tau_n) : n \in \mathbb{N}\}$ is uniformly integrable. Pass to the limit in the "discretized version" to complete the proof.

Problems = some possible examples for your notes

[1] Show that Lemma <1> also holds without the assumption that $X_t \ge 0$. Hint: Let M be a cadlag version of the martingale $\mathbb{P}\left(X_1^- \mid \mathcal{F}_t\right)$. Show that $Z_t := X_t + M_t$ is a positive superMG with cadlag paths.

[2] Suppose $\{(X_t, \mathcal{F}_t) : t \in \mathbb{R}^+\}$ is a positive supermartingale with cadlag sample paths. Suppose $\sigma_1 \le \sigma_2 \le \dots$ are stopping times taking values in $[0, \infty]$.
  (i) Show that the sequence $X_{\sigma_n}\{\sigma_n < \infty\}$ is a superMG for a suitable filtration. Compare with UGMTP Problem 6.5.
  (ii) If X is actually a positive martingale, is the sequence $X_{\sigma_n}\{\sigma_n < \infty\}$ also a martingale?
  (iii) Same question as for part (ii), except that the σ_n all take values in $[0, 1]$.

[3] Suppose $\{(X_t, \mathcal{F}_t) : t \in [0, 1]\}$ is a positive supermartingale with cadlag sample paths. Let $\mathcal{S}$ denote the set of all stopping times taking values in $[0, 1]$. Is the set of random variables $\{X_\sigma : \sigma \in \mathcal{S}\}$ uniformly integrable? It might help to track down the concept of superMGs of class [D].