Project 3

Things to explain in your notebook:

(i) How can Lévy's martingale characterization of Brownian motion be derived from a martingale central limit theorem?

(ii) What advantages are there to treating a stochastic processes with continuous sample paths as a random element of a space of continuous functions?

(iii) What is the strong Markov property for Brownian motion? Maybe sketch some sort of proof.

(iv) The completion of the filtration generated by a Brownian motion is standard.

Fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. *Negligible sets* $\mathcal{N} := \{N \in \mathcal{F} : \mathbb{P}N = 0\}$. *Assume the probability space is complete, that is, for all* $A \subseteq \Omega$, *if* $A \subseteq N \in \mathcal{N}$ *then* $A \in \mathcal{N}$. *From now on, unless indicated otherwise, also assume that all filtrations* $\{\mathcal{F}_t : t \in T\}$ *are standard, that is,* $\mathcal{N} \subseteq \mathcal{F}_t = \mathcal{F}_{t+}$ *for each* t.

<1> **Theorem.** *(McLeish 1974) For each* n *in* $\mathbb{N}$ *let* $\{\xi_{nj} : j = 0, \ldots, k_n\}$ *be a martingale difference array, with respect to a filtration* $\{\mathcal{F}_{nj}\}$, *for which:*

(i) $\sum_j \xi_{nj}^2 \to 1$ *in probability;*

(ii) $\max_j |\xi_{nj}| \to 0$ *in probability;*

(iii) $\sup_n \mathbb{P} \max_j \xi_{nj}^2 < \infty$.

Then $\sum_j \xi_{nj} \rightsquigarrow N(0,1)$ *as* $n \to \infty$.

Remark. In the last right-hand side on line 5 page 202 of UGMTP a factor X_n is missing. We need the fact that $X_n = O_p(1)$ to prove that $Y_n \to 0$ in probability.

<2> **Lévy's martingale characterization of Brownian motion.** *Suppose* $\{X_t : 0 \le t \le 1\}$ *is a martingale with continuous sample paths and* $X_0 = 0$. *Suppose also that* $X_t^2 - t$ *is a martingale. Then* X *is a Brownian motion.*

Rigorous proof that $X_1 \sim N(0,1)$. Use stopping times to cut the path into increments corresponding to the nth row of martingale differences in Theorem <1>. Omit most of the subscript n's. Take $\tau_0 = 0$ and

$$\tau_{i+1} = \min\left(n^{-1} + \tau_i, \quad \inf\{t \ge \tau_i : |X(t) - X(\tau_i)| \ge n^{-1}\right)$$

For $j = 1, 2, \ldots$ define

$$\xi_j := X(\tau_j) - X(\tau_{j-1})$$
$$\delta_j := \tau_j - \tau_{j-1}$$
$$V_j := \xi_j^2 - \delta_j$$

Write $\mathbb{P}_j(\ldots)$ for $\mathbb{P}(\ldots \mid \mathcal{F}(\tau_j))$.

- Check that $\mathbb{P}_{j-1}\xi_j = 0$ and $\mathbb{P}_{j-1}\left(V_j\right) = 0$, almost surely.
- Show that $\max_j |\xi_j| \le n^{-2}$ and $\max_j \delta_j \le n^{-1}$.
- Show that there exist $\{k_n\}$ such that $\mathbb{P}\{\sum_{j \le k_n} \delta_j \ne 1\} \to 0$ as $n \to \infty$.

- Show that $\mathbb{P}\sum_{j\le k_n} V_j = 0$ and
$$\mathbb{P}\Big(\sum_{j\le k_n} V_j\Big)^2 = \sum_{j\le k_n}\mathbb{P}V_j^2 \le n^{-1}\mathbb{P}\sum_{j\le k_n}|V_j| \le n^{-1}\mathbb{P}\sum_{j\le k_n}\big(V_j + 2\delta_j\big) \to 0 \qquad \text{as } n\to\infty.$$
- Deduce that $\sum_{j\le k_n}\xi_j^2 \to 1$ in probability.
- Deduce that $X(\tau_{k_n}) \rightsquigarrow N(0,1)$.
- Deduce that $X_1 \sim N(0,1)$.
- For enthusiasts: Extend the argument to show that X is a Brownian motion.

□

Random elements of a function space

Suppose $\{X_t : t\in\mathbb{R}^+\}$ is a process with continuous sample paths. That is, for each fixed ω the sample path $X(\cdot,\omega)$ is a member of $C[0,\infty)$, the set of all continuous real functions (not necessarily bounded) on $\mathbb{R}^+$. Equip $C[0,\infty)$ with its ***cylinder sigma-field*** $\mathcal{C}$, which is defined as the smallest sigma-field on $C[0,\infty)$ for which each coordinate map π_t is $\mathcal{C}\backslash\mathcal{B}(\mathbb{R})$-measurable. Then $\omega\mapsto X(\cdot,\omega)$ is an $\mathcal{F}\backslash\mathcal{C}$-measurable map from Ω into $C[0,\infty)$. See Problem [2].

For fixed $t\in\mathbb{R}^+$, $\pi_t(x) := x(t)$ for $x\in C[0,\infty)$.

- Find some nontrivial examples of sets in $\mathcal{C}$. For example, is the set $\{x\in C[0,\infty) : \sup_t x(t)\le 6\}$ in $\mathcal{C}$? How about the set $\{x\in C[0,\infty) : x$ has a finite derivative at 1 $\}$?

The distribution of X is a probability measure defined on $\mathcal{C}$, the image of $\mathbb{P}$ under the map $\omega\mapsto X(\cdot,\omega)$. For example, for a standard Brownian motion, the distribution is called ***Wiener measure***, which I will denote by the symbol $\mathbb{W}$. In other words, if B is a standard Brownian motion, and at least if $f : C[0,\infty)\to\mathbb{R}^+$ is a $\mathcal{C}\backslash\mathcal{B}(\mathbb{R}^+)$-measurable function, then

Note: "at least" is an invitation for you to extend the result to a larger set of functions

$$\mathbb{P}^\omega f(X(\cdot,\omega)) = \mathbb{W}^x f(x).$$

Sometimes I will slip into old-fashioned terminology and call a real-valued (or extended-real-valued) function a ***functional*** if it is defined on a space of functions.

To each stochastic process $\{X_t : t\in\mathbb{R}^+\}$ there is a ***natural filtration*** (sometimes called a raw filtration),

$$\mathcal{F}_t^\circ := \sigma\{X_s : 0\le s\le t\} \qquad \text{for } t\in\mathbb{R}^+$$

with $\mathcal{F}_\infty^\circ := \sigma\{X_s : s\in\mathbb{R}^+\}$. A generating class argument (compare with Problem [1]) shows that each $\mathcal{F}_\infty^\circ$-measurable random variable on Ω can be expressed as a composition $h(X(\cdot,\omega))$ with h a $\mathcal{C}$-measurable functional on $C[0,\infty)$. Moreover, if for each fixed $\tau\in\mathbb{R}^+$ we define the stopping operator $K_\tau : C[0,\infty)\to C[0,\infty)$ by

$$(K_\tau x)(t) = x(\tau\wedge t) \qquad \text{for } t\in\mathbb{R}^+,$$

then (Problem [4]) each $\mathcal{F}_t^\circ$-measurable random variable on Ω can be expressed as a composition $h(K_t X(\cdot,\omega))$ with h a $\mathcal{C}$-measurable functional on $C[0,\infty)$.

Decomposition of Brownian motion sample paths

Think of a standard Brownian motion $\{B_t : t\in\mathbb{R}^+\}$ as a random element of $C[0,\infty)$. For a fixed $\tau\in\mathbb{R}^+$, the process

$$Z(t) := B(\tau+t) - B(\tau) \qquad \text{for } t\in\mathbb{R}^+$$

is also a Brownian motion (with respect to which filtration?). Moreover the process Z, as a random element of $(C[0,\infty), \mathcal{C})$, is independent of $\mathcal{F}_\tau$. What does this assertion mean and how would you prove it? The assertion can be reexpressed in several useful ways.

Compare with the Brownian motion chapter of UGMTP.

You might try your skills at generating-class arguments to establish some of the following. You might also give some special cases as examples. Define the shift operator S_τ by

$$(S_\tau x)(t) = \begin{cases} 0 & \text{for } 0 \le t < \tau \\ x(t-\tau) & \text{for } t \ge \tau \end{cases}$$

Then:

(i) B has the same distribution as $K_\tau B + S_\tau \widetilde{B}$, where $\widetilde{B}$ is a new standard Brownian motion that is independent of B.

(ii) At least for each $\mathcal{C}$-measurable functional $h : C[0,\infty) \to \mathbb{R}^+$,

$$\mathbb{P}\left(h(B) \mid \mathcal{F}_\tau\right) = \mathbb{W}^x h(K_\tau B + S_\tau x) \qquad \text{almost surely.}$$

Notice that $K_\tau B$ is $\mathcal{F}_\tau$-measurable. It is unaffected by the integral with respect to $\mathbb{W}$.

(iii) For each $F \in \mathcal{F}_\tau$ and each h as in (ii),

$$\mathbb{P}Fh(B) = \mathbb{P}^\omega\left(\{\omega \in F\}\mathbb{W}^x h(K_\tau B(\cdot,\omega) + S_\tau x)\right)$$

(iv) At least for each $\mathcal{B}(\mathbb{R}) \otimes \mathcal{C}$-measurable map $f : \mathbb{R} \times C[0,\infty) \to \mathbb{R}^+$, and each $\mathcal{F}_\tau$-measurable random variable Y,

$$\mathbb{P}f(Y,B) = \mathbb{P}^\omega \mathbb{W}^x f(Y, K_\tau B + S_\tau x)$$

The ***strong Markov property*** for Brownian motion asserts that properties (i) to (iv) also hold for stopping times τ, provided we handle contributions from $\{\tau = \infty\}$ appropriately. For example, with f and Y as in (iv),

$$\mathbb{P}f(Y,B)\{\tau < \infty\} = \mathbb{P}^\omega \mathbb{W}^x f(Y, K_\tau B + S_\tau x)\{\tau < \infty\}$$

REMARK. Notice the several ways in which ω affects the sample path of $K_\tau B + S_\tau x$: at time t it takes the value

$$\begin{aligned} &B(t,\omega) && \text{if } 0 \le t < \tau(\omega) \\ &B(\tau(\omega),\omega) + x(t-\tau(\omega)) && \text{if } t \ge \tau(\omega) \end{aligned}$$

The Brownian filtration

cf. UGMTP §9.3

If we regard a Brownian motion $\{B_t : t \in \mathbb{R}^+\}$ as just a Gaussian process with continuus paths and a specific covariance structure, we need not explicitly mention the filtration. However, there is an implicit choice: the ***natural filtration*** defined by the process itself,

$$\begin{aligned} \mathcal{F}_t^\circ &:= \sigma\{B_s : 0 \le s \le t\} \\ &= \text{sigma-field generated by } K_t B \qquad \text{see Problem [4].} \end{aligned}$$

The process B is adapted to the natural filtration and $\{(B_t, \mathcal{F}_t^\circ) : 0 \le t \le 1\}$ is a Brownian motion in the sense defined by Project 2.

We augment the filtration by adding the negligible sets to the generating class,

$$\mathcal{F}_t = \sigma\left(\mathcal{F}_t^\circ \cup \mathcal{N}\right).$$

It should be easy for you to check that $\{(B_t, \mathcal{F}_t) : t \in \mathbb{R}^+\}$ is still a Brownian motion.

In fact, B is also a Brownian motion with respect to the standard filtration

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_{t+} = \cap_{s>t}\sigma\left(\mathcal{F}_t^\circ \cup \mathcal{N}\right)$$

Proof.

- Suppose $s < t$ and $F \in \widetilde{\mathcal{F}}_s$. Explain why it is enough to show that
$$\mathbb{P}Ff\left(B_t - B_s\right) = \left(\mathbb{P}F\right)\left(\mathbb{P}f(Z)\right) \qquad \text{where } Z \sim N(0, t-s)$$
for each bounded continuous f.
- Choose a sequence with $t > s_n \downarrow\downarrow s$. Show that $F \in \mathcal{F}_{s_n}$ and
$$\mathbb{P}Ff\left(B_t - B_{s_n}\right) = \left(\mathbb{P}F\right)\left(\mathbb{P}f(Z_n)\right) \qquad \text{where } Z_n \sim N(0, t-s_n).$$
- Pass to the limit.

□

<3> **Corollary.** *The filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$ is standard. That is, $\widetilde{\mathcal{F}}_t = \mathcal{F}_t = \sigma\left(\mathcal{F}_t^\circ \cup \mathcal{N}\right)$ for each t.*

Proof. Suppose $F \in \widetilde{\mathcal{F}}_t$. Then $F \in \mathcal{F}_s$ for each $s > t$. Fix one such s.

- Show there is an $F^\circ \in \mathcal{F}_s^\circ$ for which $F \Delta F^\circ \in \mathcal{N}$.
- Explain why there exists a $\{0,1\}$-valued, $\mathcal{C}$-measurable functional h on $C[0,\infty)$ for which $F^\circ = h(B)$.
- Show that
$$F = \mathbb{P}\left(F \mid \widetilde{\mathcal{F}}_t\right) = \mathbb{P}\left(h(B) \mid \widetilde{\mathcal{F}}_t\right) = \mathbb{W}^x h(K_t B + S_t x) \qquad \text{almost surely.}$$
- Explain why $\mathbb{W}^x h(K_t B + S_t x)$ is a $\mathcal{C}$-measurable function of $K_t B$ and hence it is $\mathcal{F}_t^\circ$-measurable.
- Conclude that $F \in \sigma(\mathcal{N} \cup \mathcal{F}_t^\circ) = \mathcal{F}_t$.

□

Problems

[1] (Taken from UGMTP) Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, and suppose that $\mathcal{Y}$ is equipped with a σ-field $\mathcal{B}$. Define $\mathcal{A}$ as the sigma-field of sets of the form $T^{-1}B$, with B in $\mathcal{B}$. Suppose $f \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$. Show that there exists a $\mathcal{B}\backslash\mathcal{B}[0,\infty]$-measurable function g from $\mathcal{Y}$ into $[0,\infty]$ such that $f(x) = g(T(x))$, for all x in $\mathcal{X}$, by following these steps.

(i) Show that $\mathcal{A}$ is a σ-field on $\mathcal{X}$. (It is called the σ-field generated by the map T. It is often denoted by $\sigma(T)$.)

(ii) Show that $\{f \geq i/2^n\} = T^{-1}B_{i,n}$ for some $B_{i,n}$ in $\mathcal{B}$. Define
$$f_n = 2^{-n}\sum_{i=1}^{4^n}\{f \geq i/2^n\} \qquad \text{and} \qquad g_n = 2^{-n}\sum_{i=1}^{4^n} B_{i,n}.$$
Show that $f_n(x) = g_n(T(x))$ for all x.

(iii) Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property. (Question: Why can't we define $g(y) = \lim g_n(y)$?)

[2] Let ψ be a map from $(\Omega, \mathcal{F})$ to $C[0,\infty)$.

(i) Show that ψ is $\mathcal{F}\backslash\mathcal{C}$-measurable if and only if $\pi_t \circ \psi$ is $\mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurable for each $t \in \mathbb{R}^+$.

(ii) Deduce that a stochastic process $\{X_t : t \in \mathbb{R}^+\}$ with continuous sample paths defines an $\mathcal{F}\backslash\mathcal{C}$-measurable map from Ω into $C[0,\infty)$.

[3] One metric for uniform convergence on compacta of function in $C[0,\infty)$ is defined by
$$d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n}\min\left(1, \sup_{0\leq t\leq n}|x(t) - y(t)|\right)$$

Show that the Borel sigma-field, generated by the open sets for this metric, is the same as the cylinder sigma-field $\mathcal{C}$.

[4] Suppose X is a stochastic process with sample paths in $C[0,\infty)$. For each fixed t, define $\mathcal{F}_t^\circ := \sigma\{X_s : 0 \le s \le t\}$.

(i) Show that $\mathcal{F}_t^\circ$ is the smallest sigma-field for which the map $\omega \mapsto K_t X(\cdot,\omega)$ is $\mathcal{F}_t^\circ\backslash\mathcal{C}$-measurable.

(ii) Deduce (via Problem [1]) that each $\mathcal{F}_t^\circ$-measurable random variable can be factorized as $h(K_t X(\cdot,\omega))$ for some $\mathcal{C}$-measurable functional $h : C[0,\infty) \to \overline{\mathbb{R}}$.

References

McLeish, D. L. (1974), 'Dependent central limit theorems and invariance principles', *Annals of Probability* **2**, 620–628.