PROJECT 4

Things to explain in your notebook:

(i) How to construct the isometric stochastic integral for a square integrable martingale.
(ii) What advantages are there to considering only predictable integrands?
(iii) Why does it suffice to have the Doléans measure defined only on the predictable sigma-field?

Notation and facts:

- Fixed complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Standard filtration.
- ***R-process*** = adapted process with cadlag sample paths
- ***L-process*** = adapted process with left-continuous sample paths with finite right limits

borrowed from Rogers & Williams (1987)

- $\mathcal{M}^2 = \mathcal{M}^2[0,1]$ = martingales with index set $[0,1]$, cadlag sample paths, and $\mathbb{P}M_1^2 < \infty$ ("square integrable martingales")
- $\mathcal{M}_0^2 = \mathcal{M}_0^2[0,1] = \{M \in \mathcal{M}^2[0,1] : M_0 \equiv 0\}$
- $\mathcal{H}_{\text{simple}}$ = the set of all ***simple processes*** of the form

<1> $$\sum_{i=0}^{N} h_i(\omega)\{t_i < t \le t_{i+1}\}$$

for some grid $0 = t_0 < t_1 < \ldots < t_{N+1} = 1$ and bounded, $\mathcal{F}(t_i)$-measurable random variables h_i. Note that $\mathcal{H}_{\text{simple}}$ is a subset of the set of all L-processes.

REMARK. Some authors call members of $\mathcal{H}_{\text{simple}}$ ***elementary processes***; others reserve that name for the situation where the t_i are replaced by stopping times. Dellacherie & Meyer (1982, §8.1) adopted the opposite convention.

- Abbreviate $\mathbb{P}(\ldots \mid \mathcal{F}_s)$ to $\mathbb{P}_s(\ldots)$.
- Doob's inequality: $\mathbb{P}\sup_{0\le t\le 1} M_t^2 \le 4\mathbb{P}M_1^2$ for $M \in \mathcal{M}^2[0,1]$.

cf. UGMTP Problem 6.9

Increasing processes as measures

Suppose $M \in \mathcal{M}^2$ is such that there exists an R-process A with increasing sample paths such that the process $N_t := M_t^2 - A_t$ is a martingale. Without loss of generality, $M_0 = A_0 = 0$. For example, for Brownian motion, $A_t \equiv t$.

REMARK. The existence of such an A for each M in $\mathcal{M}^2[0,1]$ will follow later from properties of stochastic integrals. See the discussion of quadratic variation.

Identify $A(\cdot,\omega)$ with a measure μ_ω on $\mathcal{B}(0,1]$ for which

$$\mu_\omega(0,t] = A(t,\omega) \qquad \text{for } 0 < t \le 1$$

Construct a measure μ on $\mathcal{B}(0,1] \otimes \mathcal{F}$ by

$$\mu g(t,\omega) = \mathbb{P}^\omega \mu_\omega^t g(t,\omega) \qquad \text{for which } g?$$

Notice that $\mu(0,1] \times \Omega = \mathbb{P}A_1 < \infty$.

- For Brownian motion, show that $\mu = \mathfrak{m} \otimes \mathbb{P}$ with $\mathfrak{m}$ = Lebesgue measure on $\mathcal{B}(0,1]$.
- For fixed $0 \le a < b \le 1$, define $\Delta N = N_b - N_a$, $\Delta M = M_b - M_a$, and $\Delta A = A_b - A_a$. Show that

$$0 = \mathbb{P}_a \Delta N = \mathbb{P}_a\left((\Delta M)^2 - \Delta A\right) \qquad \text{almost surely.}$$

- At least for each bounded, $\mathcal{F}_a$-measurable random variable h, deduce that

$$\mathbb{P}h(\omega)(\Delta M)^2 = \mathbb{P}h(\omega)\Delta A = \mathbb{P}^{\omega}\left(h(\omega)\mu_{\omega}^{t}\{a < t \le b\}\right)$$

<2> $$= \mu h(\omega)\{a < t \le b\}$$

Stochastic integral for simple processes

Suppose H is a simple process, as in <1>, and $M \in \mathcal{M}^2$. The stochastic integral is defined by

<3> $$\int_{(0,1]} H\,dM := \sum\nolimits_{i=0}^{N} h_i(\omega)\left(M(t_{i+1},\omega) - M(t_i,\omega)\right).$$

REMARK. Here I follow Rogers & Williams (1987, page 2) in excluding the lower endpoint from the range of integration. Dellacherie & Meyer (1982, §8.1) added an extra contribution from a possible jump in M at 0. With the $(0,1]$ interpretation, the definition depends only on the increments of M; with no loss of generality, we may therefore assume $M_0 \equiv 0$.

A similar awkwardness arises in defining $\int_0^t H\,dM$ if M has a jump at t. The notation does not distinguish between the integral over $(0,t)$ and the integral over $(0,t]$. I will use instead the Strasbourg notation $H \bullet M_1$ for $\int_{(0,1]} H\,dM$, with H multiplied by an explicit indicator function to modify the range of integration. For example, $\int_0^t H\,dM$ is obtained from <3> by substituting $H(s,\omega)\{0 < s \le t\}$ for H. Thus,

Check

<4> $$H \bullet M_t := \sum\nolimits_{i=0}^{N} h_i(\omega)\left(M(t\wedge t_{i+1},\omega) - M(t\wedge t_i,\omega)\right).$$

- You should check that $\mathbb{P}_t H \bullet M_1 = H \bullet M_t$ almost surely, so that $H \bullet M$ is a martingale (with cadlag paths).

<5> **Lemma.** *$\mathbb{P}\left(H \bullet M_1\right)^2 = \mu H^2$ for each $H \in \mathcal{H}_{\text{simple}}$,*

Proof. Expand the left-hand side of the asserted inequality as

$$\sum\nolimits_i \mathbb{P}h_i^2(\Delta_i M)^2 + 2\sum\nolimits_{i<j} \mathbb{P}h_i h_j \Delta_i M \Delta_j M \qquad \text{where } \Delta_i M = M(t_{i+1} - M(t_i).$$

Use the fact that $\mathbb{P}(\Delta_j M \mid \mathcal{F}(t_{j-1})) = 0$ to kill all the cross-product terms. Use equality <2> to simplify the other contributions to

$$\mu^{s,\omega}\sum\nolimits_i h_i(\omega)^2\{t_i < s \le t_{i+1}\} = \mu H^2$$

□

Extension by isometry

Think of $\mathcal{H}_{\text{simple}}$ as a subspace of $\mathcal{L}^2 = \mathcal{L}^2((0,1]\times\Omega, \mathcal{B}(0,1]\otimes\mathcal{F}_1, \mu)$. Then Lemma <5> shows that $H \mapsto H \bullet M_1$ is an isometry from a subspace of $\mathcal{L}^2$ to $\mathcal{L}^2(\Omega,\mathcal{F}_1,\mathbb{P})$. It extends to an isometry from $\overline{\mathcal{H}}_{\text{simple}}$, the $\mathcal{L}^2(\mu)$ closure of $\mathcal{H}_{\text{simple}}$ in $\mathcal{L}^2$, into $\mathcal{L}^2(\Omega,\mathcal{F}_1,\mathbb{P})$. The stochastic integral $H \bullet M_t$ is then taken to be a cadlag version of the martingale $\mathbb{P}_t H \bullet M_1$. In short, there is a linear map $H \mapsto H \bullet M$ from $\overline{\mathcal{H}}_{\text{simple}}$ to $\mathcal{M}_0^2$ for which, by Doob's inequality,

<6> $$\mathbb{P}\sup_{0\le t\le 1}|G \bullet M_t - H \bullet M_t|^2 \le 4\mathbb{P}|H \bullet M_1 - G \bullet M_1|^2 = \mu|G - H|^2$$

It is uniquely determined by the property, for all $a < b$ and $F \in \mathcal{F}_a$,

$$H \bullet M_1 = F\left(M_b - M_a\right) \qquad \text{if } H(t,\omega) = \{\omega \in F\}\{a < t \le b\}.$$

<7> **Example.** Let τ be a stopping time taking values in $[0,1]$. Define the stochastic interval

$$((0,\tau]] := \{(t,\omega) \in (0,1] \times \Omega : 0 < t \le \tau(\omega)\}$$

Let τ_n be the stopping time obtained by rounding τ up to the next integer multiple of 2^{-n}:

$$\tau_n(\omega) = \sum\nolimits_{i=1}^{2^n} t_i\{t_{i-1} < \tau(\omega) \le t_i\} \qquad \text{where } t_i = i/2^n.$$

- Show that
$$((0,\tau_n]] = \sum\nolimits_{i=1}^{2^n} \{t_{i-1} < t \le t_i\}\{\tau(\omega) > t_{i-1}\} \in \mathcal{H}_{\text{simple}}$$
and that $\mu\left(((0,\tau_n]] - ((0,\tau]]\right)^2 \to 0$.
- Conclude that $((0,\tau]] \bullet M_t = M_{t\wedge\tau}$. □

Predictable integrands

How large is $\overline{\mathcal{H}}_{\text{simple}}$? For Brownian motion, it s traditional to show (Chung & Williams 1990, Theorem 3.7) that $\overline{\mathcal{H}}_{\text{simple}}$ contains at least all the $\mathcal{B}(0,1] \times \mathcal{F}_1$-measurable, adapted processes that are square integrable for $\mathfrak{m} \times \mathbb{P}$. For other martingales, it is cleaner to work with a slightly smaller class of integrands.

<8> **Definition.** *The predictable sigma-field $\mathcal{P}$ is defined as the sigma-field on $(0,1] \times \Omega$ generated by the set of all L-processes. The space $\mathcal{H}^2(\mu)$ is defined as the set of all $\mathcal{P}$-measurable processes H on $(0,1] \times \Omega$ for which $\mu H^2 < \infty$.*

Notice that $\mathcal{H}_{\text{simple}} \subseteq \mathcal{H}^2(\mu)$ for the μ corresponding to each M in $\mathcal{M}_0^2$. In fact, a generating class argument shows that $\mathcal{H}^2(\mu)$ is the closure of $\mathcal{H}_{\text{simple}}$ in the space $\mathcal{L}^2(\,(0,1] \times \Omega, \mathcal{P}, \mu)$:

- Suppose H is a bounded, L-process. Define
$$H_n(t,\omega) := \sum\nolimits_{i=1}^{2^n} H(t_{i-1},\omega)\{t_{i-1} < t \le t_i\} \qquad \text{where } t_i = i/2^n$$
Show that $H_n \in \mathcal{H}_{\text{simple}}$ and that $H_n(t,\omega) \to H(t,\omega)$ for all (t,ω) and hence that $\mu\left(H_n - H\right)^2 \to 0$. Deduce that $H \in \overline{\mathcal{H}}_{\text{simple}}$.
- Invoke a generating class argument (such as the one given in the extract *generating-class-fns.pdf* from UGMTP) to deduce that $\overline{\mathcal{H}}_{\text{simple}}$ contains all bounded, $\mathcal{P}$-measurable processes.
- Then what?

The Doléans measure

If we intend only to extend the stochastic integral to predictable integrands, we do not need the measure μ that corresponds to the increasing process A to be defined on $\mathcal{B}(0,1] \otimes \mathcal{F}_1$: we only need it defined on $\mathcal{P}$. In fact, it is a much easier task to construct an appropriate μ on $\mathcal{P}$ directly from the submartingale $\{M_t^2 : 0 \le t \le 1\}$ without even assuming the existence of A. The measure μ is called the ***Doléans measure*** for the submartingale M^2. See the handout *Doleans.pdf* for a construction.

Moreover, there is another procedure (the dual predictable projection) for extending the Doléans measure to a "predictable measure" on $\mathcal{B}(0,1] \otimes \mathcal{F}_1$. A disintegration of this new measure then defines the process A. I'll prepare a handout describing the method.

Problems

[1] Show that the predictable sigma-field $\mathcal{P}$ on $(0,1] \times \Omega$ is generated by each of the following sets of processes:

(i) all sets $(a,b] \times F$ with $F \in \mathcal{F}_a$ and $0 \le a < b \le 1$

(ii) $\mathcal{H}_{\text{simple}}$

(iii) the set $\mathbb{C}$ of all adapted processes with continuous sample paths

(iv) all stochastic intervals $((0,\tau]]$ for stopping times τ taking values in $[0,1]$

(v) all sets $\{(t,\omega) \in (0,1] \times \Omega : X(t,\omega) = 0\}$, with $X \in \mathbb{C}$

References

Chung, K. L. & Williams, R. J. (1990), *Introduction to Stochastic Integration*, Birkhäuser, Boston.

Dellacherie, C. & Meyer, P. A. (1982), *Probabilities and Potential B: Theory of Martingales*, North-Holland, Amsterdam.

Rogers, L. C. G. & Williams, D. (1987), *Diffusions, Markov Processes, and Martingales: Itô Calculus*, Vol. 2, Wiley.