Project 4

Things to explain in your notebook:

(i) How to construct the isometric stochastic integral for a square integrable martingale.

(ii) What advantages are there to considering only predictable integrands?

(iii) Why does it suffice to have the Doléans measure defined only on the predictable sigma-field?

Notation and facts:

- Fixed complete probability space \((\Omega, \mathcal{F}, P)\). Standard filtration.
- **R-process** = adapted process with cadlag sample paths
- **L-process** = adapted process with left-continuous sample paths with finite right limits
- \(\mathcal{M}^2 = \mathcal{M}^2[0, 1] = \) martingales with index set \([0, 1]\), cadlag sample paths, and \(\mathbb{P}M_1^2 < \infty\) ("square integrable martingales")
- \(\mathcal{M}^2_0 = \mathcal{M}^2_0[0, 1] = \{M \in \mathcal{M}^2[0, 1] : M_0 \equiv 0\}\)
- \(H_{\text{simple}} = \) the set of all simple processes of the form

\[
\sum_{i=0}^{N} h_i(\omega)\{t_i < t \leq t_{i+1}\}
\]

for some grid \(0 = t_0 < t_1 < \ldots < t_{N+1} = 1\) and bounded, \(\mathcal{F}(t_i)\)-measurable random variables \(h_i\). Note that \(H_{\text{simple}}\) is a subset of the set of all L-processes.

**Remark.** Some authors call members of \(H_{\text{simple}}\) elementary processes; others reserve that name for the situation where the \(t_i\) are replaced by stopping times. Dellacherie & Meyer (1982, §8.1) adopted the opposite convention.

- Abbreviate \(P(\ldots \mid \mathcal{F}_t)\) to \(P_t(\ldots)\).
- Doob’s inequality: \(P \sup_{0 \leq t \leq 1} M_t^2 \leq 4\mathbb{P}M_1^2\) for \(M \in \mathcal{M}^2[0, 1]\).

Increasing processes as measures

Suppose \(M \in \mathcal{M}^2\) is such that there exists an R-process \(A\) with increasing sample paths such that the process \(N_t := M_t^2 - A_t\) is a martingale. Without loss of generality, \(M_0 = A_0 = 0\). For example, for Brownian motion, \(A_t \equiv t\).

**Remark.** The existence of such an \(A\) for each \(M\) in \(\mathcal{M}^2[0, 1]\) will follow later from properties of stochastic integrals. See the discussion of quadratic variation.

Identify \(A(\cdot, \omega)\) with a measure \(\mu_\omega\) on \(\mathcal{B}(0, 1]\) for which

\[
\mu_\omega(0, t] = A(t, \omega) \quad \text{for } 0 < t \leq 1
\]

Construct a measure \(\mu\) on \(\mathcal{B}(0, 1] \otimes \mathcal{F}\) by

\[
\mu g(t, \omega) = P_\omega \mu_\omega g(t, \omega) \quad \text{for which } g?
\]

Notice that \(\mu(0, 1] \times \Omega = \mathbb{P}A_1 < \infty\).

- For Brownian motion, show that \(\mu = m \otimes \mathbb{P}\) with \(m = \) Lebesgue measure on \(\mathcal{B}(0, 1]\).
- For fixed \(0 \leq a < b \leq 1\), define \(\Delta N = N_b - N_a\), \(\Delta M = M_b - M_a\), and \(\Delta A = A_b - A_a\). Show that

\[
0 = \mathbb{P}_a \Delta N = \mathbb{P}_a (\Delta M)^2 - \Delta A
\]

almost surely.
At least for each bounded, $\mathcal{F}_a$-measurable random variable $h$, deduce that
\[
P h(\omega)(\Delta M)^2 = P h(\omega) \Delta A = P^a (h(\omega) \mu^t \{ a < t \leq b \})
\]
\[
= \mu h(\omega) \{ a < t \leq b \}
\]
\[<2>\]

**Stochastic integral for simple processes**

Suppose $H$ is a simple process, as in $<1>$, and $M \in \mathcal{N}^2$. The stochastic integral is defined by
\[
\int_{(0,1]} H \, dM := \sum_{i=0}^N h_i(\omega) \left( M(t_{i+1}, \omega) - M(t_i, \omega) \right).
\]

**Remark.** Here I follow Rogers & Williams (1987, page 2) in excluding the lower endpoint from the range of integration. Dellacherie & Meyer (1982, §8.1) added an extra contribution from a possible jump in $M$ at 0. With the $(0,1]$ interpretation, the definition depends only on the increments of $M$; with no loss of generality, we may therefore assume $M_0 \equiv 0$.

A similar awkwardness arises in defining $\int_0^1 H \, dM$ if $M$ has a jump at $t$. The notation does not distinguish between the integral over $(0,t)$ and the integral over $(0,1]$. I will use instead the Strasbourg notation $H \cdot M_1$ for $\int_{(0,1]} H \, dM$, with $H$ multiplied by an explicit indicator function to modify the range of integration. For example, $\int_0^1 H \, dM$ is obtained from $<3>$ by substituting $H(s, \omega)|0 < s \leq t|$ for $H$. Thus,
\[
<4> \quad H \cdot M_t := \sum_{i=0}^N h_i(\omega) \left( M(t \wedge t_{i+1}, \omega) - M(t \wedge t_i, \omega) \right).
\]

- You should check that $P_1 H \cdot M_t = H \cdot M_t$ almost surely, so that $H \cdot M$ is a martingale (with cadlag paths).

**Lemma.** $P \left( H \cdot M_1 \right)^2 = \mu H^2$ for each $H \in \mathcal{H}_{\text{simple}}$.

**Proof.** Expand the left-hand side of the asserted inequality as
\[
\sum_i P h_i^2(\Delta_i M)^2 + 2 \sum_{i<j} P h_i h_j \Delta_i M \Delta_j M
\]
where $\Delta_i M = M(t_{i+1}) - M(t_i)$.

Use the fact that $P(\Delta_i M \mid \mathcal{F}(t_{j-1})) = 0$ to kill all the cross-product terms. Use equality $<2>$ to simplify the other contributions to
\[
\mu \sum_i h_i(\omega)^2 \{ t_i < s \leq t_{i+1} \} = \mu H^2
\]

\[\square\]

**Extension by isometry**

Think of $\mathcal{H}_{\text{simple}}$ as a subspace of $\mathcal{L}^2 = \mathcal{L}^2((0,1] \times \Omega, \mathcal{B}(0,1] \otimes \mathcal{F}_1, \mu)$. Then Lemma $<5>$ shows that $H \mapsto H \cdot M_1$ is an isometry from a subspace of $\mathcal{L}^2$ to $\mathcal{L}^2(\Omega, \mathcal{F}_1, \mathbb{P})$. It extends to an isometry from $\mathcal{F}_{\text{simple}}$, the $\mathcal{L}^2(\mu)$ closure of $\mathcal{H}_{\text{simple}}$ in $\mathcal{L}^2$, into $\mathcal{L}^2(\Omega, \mathcal{F}_1, \mathbb{P})$. The stochastic integral $H \cdot M_1$ is then taken to be a cadlag version of the martingale $\mathbb{P}_t H \cdot M_t$. In short, there is a linear map $H \mapsto H \cdot M$ from $\mathcal{F}_{\text{simple}}$ to $\mathcal{N}_0^2$ for which, by Doob’s inequality,
\[
<6> \quad P \sup_{0 \leq t \leq 1} |G \cdot M_t - H \cdot M_t|^2 \leq 4P |H \cdot M_1 - G \cdot M_1|^2 = \mu |G - H|^2
\]

It is uniquely determined by the property, for all $a < b$ and $F \in \mathcal{F}_a$,
\[
H \cdot M_1 = F (M_b - M_a) \quad \text{if} \quad H(t, \omega) = \{ \omega \in F \mid a < t \leq b \}.
\]
Example. Let $\tau$ be a stopping time taking values in $[0, 1]$. Define the stochastic interval

$$((0, \tau]) := \{(t, \omega) \in (0, 1] \times \Omega : 0 < t \leq \tau(\omega)\}$$

Let $\tau_n$ be the stopping time obtained by rounding $\tau$ up to the next integer multiple of $2^{-n}$:

$$\tau_n(\omega) = \sum_{i=1}^{2^n} t_i \{t_{i-1} < \tau(\omega) \leq t_i\} \quad \text{where} \quad t_i = i/2^n.$$  

- Show that
  $$((0, \tau_n]) = \sum_{i=1}^{2^n} \{t_{i-1} < t \leq t_i\} \tau(\omega) > t_{i-1}) \in \mathcal{H}_{\text{simple}}$$
  and that $\mu((0, \tau_n]) - ((0, \tau])^2 \to 0$.

  $\square$

- Conclude that $((0, \tau]) M_t = M_{\tau_n}$.

### Predictable integrands

How large is $\mathcal{H}_{\text{simple}}$? For Brownian motion, it is traditional to show (Chung & Williams 1990, Theorem 3.7) that $\mathcal{H}_{\text{simple}}$ contains at least all the $\mathcal{B}(0, 1] \otimes \mathcal{F}_1$-measurable, adapted processes that are square integrable for $m \times \mathbb{P}$. For other martingales, it is cleaner to work with a slightly smaller class of integrands.

Definition. The predictable sigma-field $\mathcal{P}$ is defined as the sigma-field on $(0, 1] \times \Omega$ generated by the set of all $L$-processes. The space $\mathcal{H}^2(\mu)$ is defined as the set of all $\mathcal{P}$-measurable processes $H$ on $(0, 1] \times \Omega$ for which $\mu H^2 < \infty$.

Notice that $\mathcal{H}_{\text{simple}} \subseteq \mathcal{H}^2(\mu)$ for the $\mu$ corresponding to each $M$ in $\mathcal{M}_2$. In fact, a generating class argument shows that $\mathcal{H}^2(\mu)$ is the closure of $\mathcal{H}_{\text{simple}}$ in the space $L^2((0, 1] \times \Omega, \mathcal{P}, \mu)$:

- Suppose $H$ is a bounded, $L$-process. Define
  $$H_n(t, \omega) := \sum_{i=1}^{2^n} H(t_{i-1}, \omega) \{t_{i-1} < t \leq t_i\} \quad \text{where} \quad t_i = i/2^n.$$  
  Show that $H_n \in \mathcal{H}_{\text{simple}}$ and that $H_n(t, \omega) \to H(t, \omega)$ for all $(t, \omega)$ and hence that $\mu (H_n - H)^2 \to 0$. Deduce that $H \in \mathcal{H}_{\text{simple}}$.

- Invoke a generating class argument (such as the one given in the extract generating-class-fns.pdf from UGMT) to deduce that $\mathcal{H}_{\text{simple}}$ contains all bounded, $\mathcal{P}$-measurable processes.

- Then what?

### The Doleans measure

If we intend only to extend the stochastic integral to predictable integrands, we do not need the measure $\mu$ that corresponds to the increasing process $A$ to be defined on $\mathcal{B}(0, 1] \otimes \mathcal{F}_1$; we only need it defined on $\mathcal{P}$. In fact, it is a much easier task to construct an appropriate $\mu$ on $\mathcal{P}$ directly from the submartingale $\{M_t^2 : 0 \leq t \leq 1\}$ without even assuming the existence of $A$. The measure $\mu$ is called the Doleans measure for the submartingale $M^2$. See the handout Doleans.pdf for a construction.

Moreover, there is another procedure (the dual predictable projection) for extending the Doleans measure to a “predictable measure” on $\mathcal{B}(0, 1] \otimes \mathcal{F}_1$. A disintegration of this new measure then defines the process $A$. I’ll prepare a handout describing the method.
Problems

[1] Show that the predictable sigma-field $\mathcal{P}$ on $(0, 1] \times \Omega$ is generated by each of the following sets of processes:

(i) all sets $(a, b] \times F$ with $F \in \mathcal{F}_a$ and $0 \leq a < b \leq 1$

(ii) $\mathcal{H}_{\text{simple}}$

(iii) the set $\mathcal{C}$ of all adapted processes with continuous sample paths

(iv) all stochastic intervals $((0, \tau])$ for stopping times $\tau$ taking values in $[0, 1]$

(v) all sets $\{(t, \omega) \in (0, 1] \times \Omega : X(t, \omega) = 0\}$, with $X \in \mathcal{C}$

References

