Project 5

This week I would like you to consolidate your understanding of the material from the last two weeks by working through some problems.

- Read the handout *Doleans.pdf,* at least up to Theorem 5. Try to explain the assertions flagged by the symbol $\Rrightarrow$. Try to solve Problem [2].

nonstandard notation

<1> **Definition.** *Suppose X is a process and τ is a stopping time. Define the* ***stopped process*** $X_{\wedge\tau}$ *to be the process for which* $X_{\wedge\tau}(t,\omega) = X(\tau(\omega)\wedge t, \omega)$.

Problems

[1] Suppose $M \in \mathcal{M}^2[0,1]$ has continuous sample paths.

(i) For each H in $\mathcal{H}_{\text{simple}}$, show that $H \bullet M$ has continuous sample paths.

(ii) Suppose $\{H_n : n \in \mathbb{N}\} \subseteq \mathcal{H}_{\text{simple}}$ and $\mu|H_n - H|^2 \to 0$. Use Doob's maximal inequality to show that there exists a subsequence $\mathbb{N}_1$ along which

$$\sum_{n\in\mathbb{N}_1} \mathbb{P} \sup_{0\le t\le 1} |H_n \bullet M_t - H \bullet M_t| < \infty$$

(iii) Deduce that there is a version of $H \bullet M$ with continuous sample paths.

[2] Suppose $\{\tau_n : n \in \mathbb{N}\}$ is a sequence of $[0,1]$-valued stopping times for which $\tau_n \le \tau$ and $\tau_n(\omega) \uparrow\uparrow \tau(\omega)$ at each ω for which $\tau(\omega) > 0$. Prove that $[[\tau, 1]] \in \mathcal{P}$.

Remark. The sequence $\{\tau_n\}$ is said to ***foretell*** τ. Existence of a foretelling sequence was originally used to define the concept of a ***predictable stopping time***. The modern definition requires $[[\tau, 1]] \in \mathcal{P}$. In fact, the two definitions are almost equivalent, as you will see from a later handout.

[3] Suppose $M \in \mathcal{M}_0^2[0,1]$ and $H \in \mathcal{H}^2(\mu)$, where μ is the Doléans measure defined by the submartingale M_t^2. Let τ be a $[0,1]$-valued stopping time. Let X denote the martingale $H \bullet M$.

(i) Define $N = M_{\wedge\tau}$. Show that $N \in \mathcal{M}_0^2[0,1]$.

(ii) Show that, with probability one,

$$X_{t\wedge\tau} = \Big(\big(H((0,\tau]]\big) \bullet M\Big)_t = H \bullet N_t \qquad \text{for } 0 \le t \le 1.$$

Hint: Consider first the case where $H \in \mathcal{H}_{\text{simple}}$ and τ takes values only in a finite subset of $[0,1]$. Extend to general τ by rounding up to integer multiples of 2^{-n}.

(iii) Show that the Doléans measure ν for the submartingale $(H \bullet M)_t^2$ has density H^2 with respect to μ. Hint: Remember that the Doléans measure is uniquely determined by the values it gives to the stochastic intervals $((0,\tau]]$.

(iv) Suppose $K \in \mathcal{H}^2(\nu)$. Show that $KH \in \mathcal{H}^2(\mu)$ and $K \bullet (H \bullet M) = (HK) \bullet M$.

[4] Suppose $\mu = \mathfrak{m} \otimes \mathbb{P}$, defined on $\mathcal{B}(0,1] \otimes \mathcal{F}_1$. Let $\{X_t : 0 \le t \le 1\}$ be progressively measurable.

(i) Suppose X is bounded, that is, $\sup_{t,\omega} |X(t,\omega)| < \infty$. Define

$$H_n(t,\omega) := n \int_{t-n^{-1}}^{t} X(s,\omega)\, ds$$

(How should you understand the definition when $t < n^{-1}$?) Show that H_n is predictable and that $\int_0^1 |H_n(t,\omega) - X(t,\omega)|^2\,dt \to 0$ for each ω.

(ii) Deduce that $\mu|H_n - X|^2 \to 0$.

(iii) Deduce that $X \in \overline{\mathcal{H}}_{\text{simple}}$, the closure in $\mathcal{L}^2(\mathcal{B}(0,1] \otimes \mathcal{F}_1, \mu)$, if $\mu X^2 < \infty$.

[5] Suppose $M \in \mathcal{M}_0^2[0,1]$, with μ the Doléans measure defined by the submartingale M_t^2. Suppose $\psi : \mathcal{H}^2(\mu) \to \mathcal{M}_0^2[0,1]$ is a linear map, in the sense that $\psi(\alpha H + \beta K)_t = \alpha\psi(H)_t + \beta\psi(K)_t$ almost surely, for each $t \in [0,1]$ and constants $\alpha, \beta \in \mathbb{R}$. Suppose that

(a) $\psi(((0,\tau]])_t = M_{t\wedge\tau}$ almost surely, for each stopping time τ.

(b) if $1 \geq |H_n| \to 0$ pointwise then $\psi(H_n)_t \to 0$ in probability, for each fixed t.

Show that these properties characterize the stochastic integral, in the following senses.

(i) Show that $\psi(H)_t = H \bullet M_t$ almost surely, for each t. Hint: Consider the collection of all bounded, nonnegative, predictable processes H for which $\psi(H)_t = H \bullet M_t$ almost surely, for every t. Use a generating class argument.

(ii) If, in addition, $\psi(H_n)_t \to 0$ in probability whenever $\mu H_n^2 \to 0$, show that the conclusion from part (i) also holds for every H in $\mathcal{H}_2(\mu)$.