Project 6

This week there are many small details that might occcupy your attention. I would be satisfied if you concentrated on some of the more important points. Things to explain in your notebook:

(i) Why is the theory for $\mathcal{M}_0^2(\mathbb{R}^+)$ almost the same as the theory for $\mathcal{M}_0^2[0,1]$?

(ii) Why do we get sigma-finite Doléans measures for the submartingales corresponding to $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ processes?

(iii) Why can $H \bullet M$ be built up pathwise from isometric stochastic integrals when $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ and $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$?

(iv) Why do we need to replace $\mathcal{L}^2(\mathbb{P})$ convergence by convergence in probability after localizing?

Square-integrable martingales indexed by $\mathbb{R}^+$

- Define $\mathcal{M}^2(\mathbb{R}^+)$ as the set of all square-integrable martingales, that is, cadlag martingales $\{M_t : t \in \mathbb{R}^+\}$ for which $\sup_t \mathbb{P}M_t^2 < \infty$.
- Define $\mathcal{F}_\infty = \sigma\left(\cup_{t\in\mathbb{R}^+}\mathcal{F}_t\right)$. If $M \in \mathcal{M}^2(\mathbb{R}^+)$ then there exists an $M_\infty \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ such that $M_t \to M_\infty$ almost surely and $\mathbb{P}|M_t - M_\infty|^2 \to 0$ as $t \to \infty$. Moreover, $M_t = \mathbb{P}(M_\infty \mid \mathcal{F}_t)$ almost surely.
- Note that $\{(M_t, \mathcal{F}_t) : 0 \le t \le \infty\}$ is also a martingale. Suppose ψ is a one-to-one map from $[0,1]$ onto $[0,\infty]$, such as $\psi(s) = s/(1-s)$. Define $\mathcal{G}_s := \mathcal{F}(\psi(s))$ and $N_s = M(\psi(s))$. Then $\{(N_s, \mathcal{G}_s) : 0 \le s \le 1\}$ belongs to $\mathcal{M}^2[0,1]$. All the theory for the isometric stochastic integrals with respect to $\mathcal{M}^2[0,1]$ processes carries over to analogous theory for $\mathcal{M}^2(\mathbb{R}^+)$.
- Note a subtle difference: For $\mathcal{M}^2(\mathbb{R}^+)$ we have left continuity of sample paths at ∞, by construction of M_∞. For $\mathcal{M}^2[0,1]$ we did not require left continuity at 1. Also we did not require that $\mathcal{F}_1 = \sigma\left(\cup_{t<1}\mathcal{F}_t\right)$. A better analogy would allow $\mathcal{F}_\infty$ to be larger than $\sigma\left(\cup_{t\in\mathbb{R}^+}\mathcal{F}_t\right)$ and would allow M to have a jump at ∞.

Localization

<1> **Definition.** *Suppose X is a process and τ is a stopping time. Define the* ***stopped process*** *$X_{\wedge\tau}$ to be the process for which $X_{\wedge\tau}(t,\omega) = X(\tau(\omega)\wedge t, \omega)$.*

nonstandard notation

<2> **Definition.** *Suppose $\mathcal{W}$ is a set of processes (indexed by $\mathbb{R}^+$) that is stable under stopping, $W \mapsto W_{\wedge\tau}$. Say that a process X is locally in $\mathcal{W}$ if there exists a sequence of stopping times $\{\tau_k\}$ with $\tau_k \uparrow \infty$ and $X_{\wedge\tau_k} \in \mathcal{W}$ for each k. Call $\{\tau_k\}$ a $\mathcal{W}$-**localizing sequence** for X. Write $\text{loc}\mathcal{W}$ for the set of all processes that are locally in $\mathcal{W}$.*

or reducing sequence

Remark. Notice that if $\{\tau_k\}$ is a $\mathcal{W}$-localizing sequence for X then so is $\{k \wedge \tau_k\}$. Thus we can always require each τ_k in a localizing sequence to be a bounded stopping time.

Predictable sigma-field

The predictable sigma-field $\mathcal{P}$ on $(0,\infty)\times\Omega$ is again defined as the sigma-field generated by all L-processes.

Remark. For $(0,1]\times\Omega$ the predictable sigma-field contains some subsets of $\{1\}\times\Omega$. For $(0,\infty)\times\Omega$, subsets of $\{\infty\}\times\Omega$ are not in $\mathcal{P}$. Maybe it would be better to define $\mathcal{P}$ on $(0,\infty]\times\Omega$.

Stochastic intervals

For stopping times σ and τ taking values in $\mathbb{R}^+ \cup \{\infty\}$ define

$$((\sigma, \tau]] := \{(t, \omega) \in \mathbb{R}^+ \times \Omega : \sigma(\omega) < t \leq \tau(\omega)\},$$

and so on. Note well that the stochastic interval is a subset of $\mathbb{R}^+ \times \Omega$. Points (t, ω) with $t = \infty$ are not included, even at ω for which $\tau(\omega) = \infty$. In particular, for $\sigma \equiv 0$ and $\tau \equiv \infty$ we get

$$((0, \infty]] = \mathbb{R}^+ \times \Omega.$$

Don't be misled by the "$\infty]]$" into assuming that $\{\infty\} \times \Omega$ is included.

REMARK. The convention that ∞ is excluded makes possible some neat arguments, even though it spoils the analogy with stochastic subintervals of $(0, 1] \times \Omega$. Although sorely tempted to buck tradition, I decided to stick with established usage for fear of unwanted exceptions to established theorems.

- Write $\mathcal{T}$ for the set of all $[0, \infty]$-valued stopping times. Is it true that $\mathcal{P}$ is generated by the set of all stochastic intervals $((0, \tau]]$ for $\tau \in \mathcal{T}$?
- If $M \in \mathcal{M}_0^2(\mathbb{R}^+)$ explain why there exists a finite, countably-additive measure on $\mathcal{P}$ (the Doléans measure for the submartingale M^2) for which

  $$\mu(a, b] \times F = \mathbb{P}F(M_b - M_a)^2 \qquad \text{for } F \in \mathcal{F}_a, \text{ and } 0 \leq a < b < \infty.$$

  Could we also allow $b = \infty$? Is it still true that

  $$\mu((0, \tau]] = \mathbb{P}M_\tau^2 \qquad \text{for each } \tau \in \mathcal{T}?$$

  How should the last equality be interpreted when $\{\omega : \tau(\omega) = \infty\} \neq \emptyset$?

Locally square-integrable martingales

- Consider first the case of a process M for which there exists a stopping time σ such that $N := M_{\wedge\sigma} \in \mathcal{M}^2(\mathbb{R}^+)$. Let μ be the Doléans measure on $\mathcal{P}$ for the square-integrable submartingale N^2.
  (i) Is it true that $N_\infty = M_\sigma$? What would this equality be asserting about those ω at which $\sigma(\omega) = \infty$?
  (ii) Show that $\mu((0, \infty]] = \sup_t \mathbb{P}M_{t\wedge\sigma}^2 = \mathbb{P}N_\infty^2$.
  (iii) Show that $\mu((t \wedge \sigma, \infty]] = \mathbb{P}(N_\infty - M_{t\wedge\sigma})^2 \to 0$ as $t \to \infty$.
  (iv) Conclude that μ is a finite measure that concentrates all its mass on the stochastic interval $((0, \sigma]]$.
- Now suppose $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$, with localizing sequence $\{\tau_k : k \in \mathbb{N}\}$. Write μ_k for the Doléans measure of the submartingale $M_{\wedge\tau_k}^2$.
  (i) Show that μ_k is a finite measure concentrating on $((0, \tau_k]]$ and that the restriction of μ_{k+1} to $((0, \tau_k]]$ equals μ_k.
  (ii) Define μ on $\mathcal{P}$ by $\mu H := \sup_{k\in\mathbb{N}} \mu_k H$. Show that μ is a sigma-finite, countably-additive measure for which

  $$\mu((0, \tau]] = \sup_k \mathbb{P}M_{\tau\wedge\tau_k}^2 \qquad \text{for all } \tau \in \mathcal{T}.$$

  (iii) Suppose $\{\sigma_k : k \in \mathbb{N}\}$ is another localizing sequence for M. Show that

  $$\mu((0, \tau]] = \sup_k \mathbb{P}M_{\tau\wedge\sigma_k}^2 \qquad \text{for all } \tau \in \mathcal{T}.$$

  That is, show that μ does not depend on the choice of localizing sequence for M.

Locally bounded predictable processes

Write $\mathcal{H}_{\text{Bdd}}$ for the set of all bounded, $\mathcal{P}$-measurable processes.

- Show that every L-process X with $\sup_\omega |X_0(\omega)| < \infty$ belongs to $\text{loc}\mathcal{H}_{\text{Bdd}}$. Hint: Consider $\tau_k(\omega) := \inf\{t \in \mathbb{R}^+ : |X_t(\omega)| \geq k\}$.

  REMARK. Does an L-process have time set $[0, \infty)$ or $(0, \infty)$? Perhaps the assertion would be better expressed as: the restriction of X to $(0, \infty) \times \Omega$ belongs to $\text{loc}\mathcal{H}_{\text{Bdd}}$. In that case, the assumption about X_0 is superfluous. D&M have some delicate conventions and definitions for handling contributions from $\{0\} \times \Omega$.

- (Much harder) Is the previous assertion still true if we replace L-processes by $\mathcal{P}$-measurable processes? What if we also require each sample path to be cadlag?

  REMARK. A complete resolution of this question requires some facts about predictable stopping times and predictable cross-sections. Compare with Métivier (1982, Section 6).

Localization of the isometric stochastic integral

The new stochastic integral will be defined indirectly by a sequence of isometries. The continuity properties of $H \bullet M$ will be expressed not via $\mathcal{L}^2$ bounds but by means of the concept of ***uniform convergence in probability on compact intervals***. For a sequence of processes $\{Z_n\}$, write $Z_n \xrightarrow{ucpc} Z$ to mean that $\sup_{0 \leq s \leq t} |Z_n(s, \omega) - Z(s, \omega)| \to 0$ in probability, for each t in $\mathbb{R}^+$.

nonstandard notation

<3> **Theorem.** *Suppose $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$. There exists a linear map $H \mapsto H \bullet M$ from $\text{loc}\mathcal{H}_{\text{Bdd}}$ into $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ with the following properties.*

(i) $((0, \tau]] \bullet M_t = M_{t\wedge\tau}$ *for all* $\tau \in \mathcal{T}$.

(ii) $(H \bullet M)_{t\wedge\tau} = \big(H((0, \tau]]\big) \bullet M_t = \big(H \bullet M_{\wedge\tau}\big)_t$, *for all* $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ *and all* $\tau \in \mathcal{T}$.

(iii) If M has continuous sample paths then so does $H \bullet M$.

*(iv) Suppose $\{H^{(n)} : n \in \mathbb{N}\} \subseteq \text{loc}\mathcal{H}_{\text{Bdd}}$ and $H^{(n)}(t, \omega) \to 0$ for each (t, ω). Suppose that the sequence is **locally uniformly bounded**: there exist stopping times with $\tau_k \uparrow \infty$ and finite constants C_k such that $|H^{(n)}_{\wedge\tau_k}| \leq C_k$ for each n and each k. Then $H^{(n)} \bullet M \xrightarrow{ucpc} 0$.*

(v) $K \bullet (H \bullet M) = (KH) \bullet M$ *for* $K, H \in \text{loc}\mathcal{H}_{\text{Bdd}}$.

Sketch of a proof. Suppose M has localizing sequence $\{\tau_k : k \in \mathbb{N}\}$ and $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ has localizing sequence $\{\sigma_k : k \in \mathbb{N}\}$.

- Why is there no loss of generality in assuming that $\sigma_k = \tau_k$ for every k?
- Write $M^{(k)}$ for $M_{\wedge\tau_k}$. Define $X^{(k)}$ to be the square integrable martingale

$$X^{(k)} = H_{\wedge\tau_k} \bullet M^{(k)} = \big(H((0, \tau_k]]\big) \bullet M^{(k)}.$$

  Why are the two integrals the same, up to some sort of almost sure equivalence?

- Show that $X^{(k)}(t, \omega) = X^{(k)}(t \wedge \tau_k(\omega), \omega)$ for all $t \in \mathbb{R}^+$. That is, show that the sample paths are constant for $t \geq \tau_k(\omega)$. Do we need some sort of almost sure qualification here?
- Show that, on a set of ω with probability one,

$$X^{(k+1)}(t \wedge \tau_k(\omega), \omega) = X^{(k)}(t \wedge \tau_k(\omega), \omega) \qquad \text{for all } t \in \mathbb{R}^+.$$

- Show that there is an R-process X for which, on a set of ω with probability one,

$$X(t \wedge \tau_k(\omega), \omega) = X^{(k)}(t \wedge \tau_k(\omega), \omega) \qquad \text{for all } t \in \mathbb{R}^+, \text{ all } k.$$

- Show that $X \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$, with localizing sequence $\{\tau_k : k \in \mathbb{N}\}$.
- Define $H \bullet M := X$.
- In order to establish linearity of $H \mapsto H \bullet M$, we need to show that the definition does not depend on the particular choice of the localizing sequence. (If we can use a single localizing sequence for two different H processes then linearity for the approximating $X^{(k)}$ processes will transfer to the X process.)
- For assertion (iv), we may also assume that $\{\tau_k\}$ localizes M to $\mathcal{M}_0^2(\mathbb{R}^+)$. Write μ_k for the Doléans measure of the submartingale $(M^{(k)})^2$. Then, for each fixed k, we have

$$\begin{aligned}
\mathbb{P}\sup_{s\le t}\left(H^{(n)}\bullet M\right)^2_{s\wedge\tau_k} &= \mathbb{P}\sup_{s\le t}\left(H^{(n)}((0,\tau_k]]\bullet M^{(k)}\right)^2_s && \text{by construction}\\
&\le 4\mathbb{P}\left(H^{(n)}((0,\tau_k]]\bullet M^{(k)}\right)^2_t && \text{by Doob's inequality}\\
&= 4\mu_k\left((H^{(n)})^2((0,\tau_k\wedge t]]\right)\\
&\to 0 \qquad \text{as } n\to\infty, \text{ by Dominated Convergenece.}
\end{aligned}$$

When $\tau_k > t$, which happens with probability tending to one, the processes $H^{(n)} \bullet M_{s\wedge\tau_k}$ and $H^{(n)} \bullet M_s$ coincide for all $s \le t$. The uniform convergence in probability follows.

□

Characterization of the stochastic integral

<4> **Theorem.** *Suppose $M \in \mathcal{M}_0^2(\mathbb{R}^+)$. Suppose also that $\psi : \text{loc}\mathcal{H}_{\text{Bdd}} \to \mathcal{M}_0^2(\mathbb{R}^+)$ is a linear map (in the sense of almost sure equivalence) for which*

better just to state equality for bounded τ?

(i) $((0,\tau]] \bullet M_t = M_{\tau\wedge t}$ *almost surely, for each $t \in \mathbb{R}^+$ and $\tau \in \mathcal{T}$*

(ii) If $\{H^{(n)} : n \in \mathbb{N}\} \subseteq \text{loc}\mathcal{H}_{\text{Bdd}}$ is locally uniformly bounded and $H^{(n)}(t,\omega) \to 0$ for each (t,ω) then $\psi(H^{(n)})_t \to 0$ in probability for each t.

Then $\psi(H)_t = H \bullet M_t$ almost surely for each $t \in \mathbb{R}^+$ and each $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$.

REMARK. The assertion of the Theorem can also be written: there exists a set Ω_0 with $\Omega_0^c \in \mathcal{N}$ such that

$$\psi(H)(t,\omega) = H \bullet M(t,\omega) \qquad \text{for every } t \text{ if } \omega \in \Omega_0$$

Cadlag sample paths allow us to deduce equality of whole paths from equality ot a countable dense set of times.

References

Métivier, M. (1982), *Semimartingales: A Course on Stochastic Processes*, De Gruyter, Berlin.