PROJECT 7

This week you should concentrate on understanding Theorem <4>, which states the basic properties of integrals with respect to semimartingales. The facts about finite variation are mostly for background information; you could safely regard an $\mathcal{FV}$-process to be defined as a difference of two increasing R-processes.

The facts about quadratic variation process will be used in the next Project to establblsh the Itô formula. You might prefer to postpone your careful study of $[X, Y]$ until that Project.

I do not expect you to work every Problem.

1. Cadlag functions of bounded variation

Suppose f is a real function defined on $\mathbb{R}^+$. For each finite grid

$$\mathbb{G}: \quad a = t_0 < t_1 < \ldots < t_N = b$$

on $[a, b]$ define the variation of f over the grid to be

$$V_f(\mathbb{G}, [a, b]) := \sum\nolimits_{i=1}^{N} |f(t_i) - f(t_{i-1})|$$

Say that f is of ***bounded variation*** on the interval $[a, b]$ if there exists a finite constant $V_f[a, b]$ for which

$$\sup\nolimits_{\mathbb{G}} V_f(\mathbb{G}, [a, b]) \le V_f[a, b]$$

where the supremum is taken over the set of all finite grids $\mathbb{G}$ on $[a, b]$. Say that f is of ***finite variation*** if it is of bounded variation on each bounded interal $[0, b]$.

Problems [1] and [2] establish the following facts about finite variation. Every difference $f = f_1 - f_2$ of two increasing functions is of finite variation. Conversely, if f is of finite variation then the functions $t \mapsto V_f[0, t]$ and $t \mapsto V_f[0, t] - f(t)$ are both increasing and

$$f(t) = V_f[0, t] - \big(V_f[0, t] - f(t)\big),$$

a difference of two increasing functions. Moreover, if f is cadlag then $V_f[0, t]$ is also cadlag.

2. Processes of finite variation as random (signed) measures

Notation: $L_t(\omega) = L(t, \omega)$.

Let $\{L_t : t \in \mathbb{R}^+\}$ be an R-process with increasing sample paths, adapted to a standard filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$. For each ω, the function $L(\cdot, \omega)$ defines a measure on $\mathcal{B}(\mathbb{R}^+)$,

$$\lambda_\omega[0, t] = L_t(\omega) \qquad \text{for } t \in \mathbb{R}^+.$$

The family $\Lambda = \{\lambda_\omega : \omega \in \Omega\}$ may be thought of as a ***random measure***, that is, a map from Ω into the space of (sigma-finite) measures on $\mathcal{B}(\mathbb{R}^+)$.

Notice that $\lambda_\omega\{0\} = L_0(\omega)$, an atom at the origin, which can be awkward. It will be convenient if $L_0 \equiv 0$, ensuring that λ_ω concentrates on $(0, \infty)$.

<1> **Definition.** *Write $\mathcal{FV}$, or $\mathcal{FV}(\mathbb{R}^+)$ if there is any ambiguity about the time set, for the set of all R-processes with sample paths that are of finite variation on $\mathbb{R}^+$. Write $\mathcal{FV}_0$ for the subset of $\mathcal{FV}$-processes, A, with $A_0 \equiv 0$.*

- Show that $\mathcal{FV}$ could also be defined as the set of processes expressible as a difference $A(\cdot,\omega) = L'(\cdot,\omega) - L''(\cdot,\omega)$ of two increasing R-processes.

The stochastic integral with respect to A will be defined as a difference of stochastic integrals with respect to L' and L''. Questions of uniqueness—lack of dependence on the choice of the two increasing processes—will be subsumed in the the uniqueness assertion for semimartingales.

The case where L is an increasing R-process with $L_0 \equiv 0$ will bring out the main ideas. I will leave to you the task of extending the results to a difference of two such processes. Define the stochastic integral with respect to L pathwise,

$$H \bullet L_t := \lambda_\omega^s \left(\{0 < s \le t\} H(s,\omega)\right).$$

This integral is well defined if $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$.

Indeed, suppose $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$. There exist stopping times $\tau_k \uparrow \infty$ and finite constants C_k for which $|H_{\wedge\tau_k}| \le C_k$. For each fixed ω, the function $s \mapsto H(s,\omega)$ is measurable (by Fubini, because predictable implies progressively measurable). Also $\sup_{s\le t} |H(s,\omega)| \le C_k$ when $t \le \tau_k(\omega)$. The function $H(\cdot,\omega)$ is integrable with respect to λ_ω on each bounded interval. Moreover, we have a simple bound for the contributions from the positive and negative parts of H to the stochastic integral:

$$0 \le H^\pm \bullet L_{t\wedge\tau_k} = \lambda_\omega^s \left(\{0 \le s \le t \wedge \tau_k\} H^\pm(s,\omega)\right) \le C_k L(t,\omega).$$

That is, $H^\pm \bullet L_t \le C_k L(t,\omega)$ when $t \le \tau_k(\omega)$.

- Show that the sample paths of $H^\pm \bullet L$ are cadlag and adapted. Deduce that $H \bullet L \in \mathcal{FV}_0$.

You should now be able to prove the following result by using standard facts about measures.

<2> **Theorem.** *Suppose $A \in \mathcal{FV}_0$. There is a map $H \mapsto H \bullet A$ from $\text{loc}\mathcal{H}_{\text{Bdd}}$ to $\mathcal{FV}_0$ that is linear (in the almost sure sense?) for which:*

(i) $((0,\tau]] \bullet A_t = A_{t\wedge\tau}$ *for each* $\tau \in \mathcal{T}$ *and* $t \in \mathbb{R}^+$.

(ii) $(H \bullet A)_{t\wedge\tau} = (H((0,\tau]]) \bullet A_t = H \bullet (A_{\wedge\tau})_t$ *for each* $\tau \in \mathcal{T}$ *and* $t \in \mathbb{R}^+$.

(iii) If a sequence $\{H_n\}$ *in* $\text{loc}\mathcal{H}_{\text{Bdd}}$ *is locally uniformly bounded and converges pointwise (in t and ω) to* 0 *then* $H_n \bullet A \xrightarrow{ucpc} 0$.

As you can see, there is really not much subtlety beyond the usual measure theory in the construction of stochastic integrals with respect to $\mathcal{FV}$-processes.

REMARK. The integral $H \bullet L_t$ can be defined even for processes that are not predictable or locally bounded. In fact, as there are no martingales involved in the construction, predictability is irrelevant. However, functions in $\text{loc}\mathcal{H}_{\text{Bdd}}$ will have stochastic integrals defined for both $\mathcal{FV}_0$-processes and $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$-processes.

3. Stochastic integrals with respect to semimartingales

By combining the results from the previous Section with results from Project 6, we arrive at a most satisfactory definition of the stochastic integral for a very broad class of processes.

$\mathcal{SMG}$ is nonstandard notation

<3> **Definition.** *An R-process X is called a* ***semimartingale****, for a given standard filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$, if it can be decomposed as $X_t = X_0 + M_t + A_t$ with $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ and $A \in \mathcal{FV}_0$. Write $\mathcal{SMG}$ for the class of all semimartingales and $\mathcal{SMG}_0$ for those semimartinagles with $X_0 \equiv 0$.*

Notice that $\mathcal{SMG}_0$ is stable under stopping. Moreover, every local semimartingale is a semimartingale, a fact that is surprisingly difficult (Dellacherie & Meyer 1982, §VII.26) to establish directly.

The stochastic integral $H \bullet X$ is defined as the sum of the stochastic integrals with respect to the components M and A. The value X_0 plays no role in this definition, so we may as well assume $X \in \mathcal{SMG}_0$. The resulting integral inherits the properties shared by integrals with respect to $\mathcal{FV}_0$ and integrals with respect to $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$.

linear in thealmost sure sense

<4> **Theorem.** *For each X in $\mathcal{SMG}_0$, there is a linear map $H \mapsto H \bullet X$ from $\text{loc}\mathcal{H}_{\text{Bdd}}$ into $\mathcal{SMG}_0$ such that:*

(i) $((0,\tau]] \bullet X_t = X_{t\wedge\tau}$ *for each $\tau \in \mathcal{T}$ and $t \in \mathbb{R}^+$.*

(ii) $H \bullet X_{t\wedge\tau} = (H((0,\tau]]) \bullet X_t = H \bullet (X_{\wedge\tau})_t$ *for each $\tau \in \mathcal{T}$ and $t \in \mathbb{R}^+$.*

(iii) If a sequence $\{H_n\}$ in $\text{loc}\mathcal{H}_{\text{Bdd}}$ is locally uniformly bounded and converges pointwise (in t and ω) to 0 then $H_n \bullet X \xrightarrow{ucpc} 0$.

Conversely, let ψ be another linear map from $\text{loc}\mathcal{H}_{\text{Bdd}}$ into the set of R-processes having at least the weaker properties:

(iv) $\psi(((0,\tau]])_t = X_{t\wedge\tau}$ *almost surely, for each $\tau \in \mathcal{T}$ and $t \in \mathbb{R}^+$.*

(vi) If a sequence $\{H_n\}$ in $\text{loc}\mathcal{H}_{\text{Bdd}}$ is locally uniformly bounded and converges pointwise (in t and ω) to 0 then $\psi(H_n)_t \to 0$ in probability, for each fixed t.

Then $\psi(H)_t = H \bullet X_t$ almost surely for every t.

REMARKS. The converse shows, in particular, that the stochastic integral $H \bullet X$ does not depend on the choice of the processes M and A in the semimartingale decomposition of X.

In general, I say that two processes X and Y are equal for ***almost all paths*** if $\mathbb{P}\{\exists t : X_t(\omega) \neq Y_t(\omega)\} = 0$. For processes with cadlag sample paths, this property is equivalent to $\mathbb{P}\{\omega : X_t(\omega) \neq Y_t(\omega)\} = 0$ for each t.

Outline of the proof of the converse. Define

$$\mathcal{H} := \{H \in \mathcal{H}_{\text{Bdd}} : \psi(H)_t = H \bullet X_t \text{ almost surely, for each } t \in \mathbb{R}^+\}$$

- Show that $((0,\tau]] \in \mathcal{H}$, for each $\tau \in \mathcal{T}$.
- Show that $\mathcal{H}$ is a λ-space. Hint: If $H_n \in \mathcal{H}$ and $H_n \uparrow H$, with H bounded, apply (iii) and (vi) to the uniformly bounded sequence $H - H_n$.
- Deduce that $\mathcal{H}$ equals $\mathcal{H}_{\text{Bdd}}$.
- Extend the conclusion to $\text{loc}\mathcal{H}_{\text{Bdd}}$. Hint: If $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$, with $|H_{\wedge\tau_k}| \le C_k$ for stopping times $\tau_k \uparrow \infty$, show that the processes $H_n := H((0,\tau_n]]$ are locally uniformly bounded and converge pointwise to H. □

I have found the properties of the stochastic integral asserted by the Theorem to be adequate for many arguments. I consider it a mark of defeat if I have to argue separately for the $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ and $\mathcal{FV}_0$ cases to establish a general result about semimartingales. You might try Problem [3] or [4] for practice.

The class of semimartingales is quite large. It is stable under sums (not surprising) and products (very surprising—see the next Section) and under exotic things like change of measure (to be discussed in a later Project). Even more surprisingly, semimartingales are the natural class of integrators for stochastic integrals; they are the unexpected final product of a long sequence of ad hoc improvements. You might consult Protter (1990, pages 44; 87–88; 114), who expounded the whole theory by starting from plausible linearity and continuity assumptions then working towards the conclusion that only semimartingales can have the desired properties.

Characterization due to Dellacherie? Meyer? Bichteler? Métivier? Check history.

4. Quadratic variation

In the proof of Lévy's martingale characterization of Brownian Motion, you saw how a sum of squares of increments of Brownian motion, taken over a partition via stopping times of an interval $[0, t]$, converges in probability to t. In fact, if one allows random limits, the behaviour is a general property of semimartingales. The limit is called the ***quadratic variation process*** of the semimartingale.

It is easiest to establish existence of the quadratic variation by means of an indirect stochastic integral argument. Suppose X is an R-processes with $X_0 \equiv 0$. Define the left-limit process $X_t^{\ominus} := X(t-, \omega) := \lim_{s \uparrow\uparrow t} X(s, \omega)$. (Do we need to define $X_0^{\ominus}$?)

Awkward and nonstandard notation, $X^{\ominus}$, but I want X^- for the negative part of X.

- Show that $X^{\ominus} \in \text{loc}\mathcal{H}_{\text{Bdd}}$.

<5> **Definition.** *The quadratic variation process of an X in $\mathcal{SMG}_0$ is defined as $[X, X]_t := X_t^2 - 2(X^{\ominus} \bullet X)_t$ for $t \in \mathbb{R}^+$. For general $Z \in \mathcal{SMG}$, define $[Z, Z] := [X, X]$ where $X_t := Z_t - Z_0$.*

The logic behind the name *quadratic variation* and one of the main reasons for why it is a useful process both appear in the next Theorem. The first assertion of the Theorem could even be used to define quadratic variation, but then we would have to work harder to prove existence of the limit (as for the quadratic variation of Brownian motion).

<6> **Definition.** *A **random grid** $\mathbb{G}$ is defined by a finite sequence of finite stopping times $0 \le \tau_0 \le \tau_1 \le \ldots \le \tau_k$. The mesh of the grid is defined as* $\text{mesh}(\mathbb{G}) := \max_i |\tau_{i+1} - \tau_i|$*; the max of the grid is defined as* $\max(\mathbb{G}) := \tau_k$.

To avoid double subscripting, let me write $\sum_{\mathbb{G}}$ to mean a sum taken over the stopping times that make up $\mathbb{G}$.

<7> **Theorem.** *Suppose $X \in \mathcal{SMG}_0$ and $\{\mathbb{G}_n\}$ is a sequence of random grids with* $\text{mesh}(\mathbb{G}_n) \xrightarrow{a.s.} 0$ *and* $\max(\mathbb{G}_n) \xrightarrow{a.s.} \infty$. *Then:*

(i) $\sum_{\mathbb{G}_n} \left(X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}\right)^2 \xrightarrow{ucpc} [X, X]_t$.

(ii) The process $[X, X]$ has increasing sample paths;

(iii) If τ is a stopping time then $[X_{\wedge\tau}, X_{\wedge\tau}] = [X, X]_{\wedge\tau}$.

Mention jumps as well?

Outline of proof. Without loss of generality suppose $X_0 \equiv 0$. Consider first a fixed t and a fixed grid $\mathbb{G}$: $0 = \tau_0 \le \tau_1 \le \ldots \le \tau_k$.

- Define a left-continuous process $H_{\mathbb{G}} = \sum_{\mathbb{G}} X_{\tau_i}((\tau_i, \tau_{i+1}]]$. Show that $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ and
$$H_{\mathbb{G}} \bullet X_t = \sum_{\mathbb{G}} X_{\tau_i} \left(X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}\right)$$
Hint: Look at Problem [3].
- Except on a negligible set of paths (which I will ignore for the rest of the proof), show that $H_{\mathbb{G}}$ converges pointwise to the left-limit process $X^{\ominus}$ as $\text{mesh}(\mathbb{G}) \to 0$ and $\max(\mathbb{G}) \to \infty$. Show also that $\{H_{\mathbb{G}}\}$ is locally uniformly bounded. Hint: Consider stopping times $\sigma_k := \inf\{s : |X_s| \ge k\}$.

- Abuse notation by writing $\Delta_i X$ for $X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}$. Invoke the continuity property of the stochastic integral, along a sequence of grids with $\text{mesh}(\mathbb{G}_n) \to 0$ and $\max(\mathbb{G}_n) \to \infty$, to deduce that

$$\sum\nolimits_{\mathbb{G}_n} X_{\tau_i}(\Delta_i X) = H_{\mathbb{G}_n} \bullet X_t \overset{ucpc}{\longrightarrow} X^{\ominus} \bullet X_t$$

- Show that

$$2H_{\mathbb{G}_n} \bullet X_t + \sum\nolimits_{\mathbb{G}_n} (\Delta_i X)^2 = X^2_{t\wedge\tau_k} \overset{ucpc}{\longrightarrow} X_t^2.$$

- Complete the proof of (i).
- Establish (ii) by taking the limit along a sequence of grids (deterministic grids would suffice) for which both s and t are always grid points. Note: The sums of squared increments that converge to $[X,X]_t$ will always contain extra terms in addition to those for sums converging to $[X,X]_s$.
- For assertion (iii), merely note that $\tau \wedge t$ is one of the points in the interval $[0,t]$ over which the convergence in probability is uniform. Thus

$$\sum\nolimits_{\mathbb{G}_n} \left(X_{t\wedge\tau_{i+1}\wedge\tau} - X_{t\wedge\tau_i\wedge\tau}\right)^2 \overset{\mathbb{P}}{\longrightarrow} [X,X]_{t\wedge\tau}.$$

Interpret the left-hand side as an approximating sum of squares for $[X_{\wedge\tau}, X_{\wedge\tau}]_t$. □

<8> **Corollary.** *The square of a semimartingale X is a semimartingale.*

Proof. Let $Z_t := X_t - X_0 = M_t + A_t$. Rearrange the definition of the square bracket process, $Z_t^2 = 2(Z^{\ominus} \bullet Z)_t + [Z,Z]_t$, to express Z_t^2 as a sum of a semimartingale and an increasing process. The process X_t^2 expands to $Z_t^2 + 2X_0M_t + (2X_0A_t + X_0^2)$.

- Show that the middle term is reduced to $\mathcal{M}_0^2(\mathbb{R}^+)$ by the stopping times $\tau_k \wedge \sigma_k$, where $\{\tau_k\}$ reduces M and $\sigma_k := 0\{|X_0| > k\} + \infty\{|X_0| \le k\}$. □

<9> **Corollary.** *The product of two semimartingales is a semimartingale.*

- Use the ***polarization identity***, $4XY = (X+Y)^2 - (X-Y)^2$, and the fact that sums of semimartingales are semimartingales, to reduce the assertion to the previous Corollary.

<10> **Definition.** *The* ***square bracket process*** $[X,Y]$ *of two semimartingales X and Y (also known as the quadratic covariation process of X and Y) is defined, by polarization, as*

$$4[X,Y] := [X+Y, X+Y] - [X-Y, X-Y].$$

If $X_0 \equiv 0$ and $Y_0 \equiv 0$ then $4[X,Y]_t$ equals

$$(X_t+Y_t)^2 - (X_t-Y_t)^2 - 2(X+Y)^{\ominus} \bullet (X+Y)_t + 2(X-Y)^{\ominus} \bullet (X-Y)_t$$

<11>
$$= 4X_tY_t - 4X^{\ominus} \bullet Y_t - 4Y^{\ominus} \bullet X_t.$$

REMARK. Notice that $[X,Y]$ is equal to the quadratic variation process $[X,X]$ when $X \equiv Y$. Notice also that $[X,Y] \in \mathcal{FV}_0$, being a difference of two increasing processes started at 0.

The square bracket process inherits many properties from the quadratic variation. For example, you might prove that a polarization argument derives the following result from Theorem <7>.

<12> **Theorem.** *Let X and Y be semimartingales, and $\{\mathbb{G}_n\}$ be a sequence of random grids with* $\text{mesh}(\mathbb{G}_n) \xrightarrow{a.s.} 0$ *and* $\max(\mathbb{G}_n) \xrightarrow{a.s.} \infty$. *Then*

<13> $$\sum\nolimits_{\mathbb{G}_n} \left(X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}\right)\left(Y_{t\wedge\tau_{i+1}} - Y_{t\wedge\tau_i}\right) \xrightarrow{ucp} [X, Y]_t,$$

and $[X_{\wedge\tau}, Y_{\wedge\tau}] = [X_{\wedge\tau}, Y] = [X, Y_{\wedge\tau}] = [X, Y]_{\wedge\tau}$ *for each stopping time* τ,

Problems

[1] Suppose $f = f_1 - f_2$, where f_1 and f_2 are increasing functions on $\mathbb{R}^+$. Show that

$$V_f[0, b] \le V_{f_1}[0, b] + V_{f_2}[0, b] = f_1(b) - f_1(0) + f_2(b) - f_2(0).$$

Deduce that f is of finite variation.

[2] Suppose f is a function on $\mathbb{R}^+$ with finite variation, in the sense of Section 1. Temorarily drop the subscript f on the variation functions.

(i) Suppose $\mathbb{G}$ is a grid on $[a, b]$ and that s is point of (a, b) that is not already a grid point. Show that $V(\mathbb{G}, [a, b])$ is increased if we add s as a new grid point.

(ii) Show that $V[0, a] + V[a, b] = V[0, b]$ for all $a < b$. Deduce that $t \mapsto V[0, t]$ is an increasing function

(iii) Suppose $0 < s < t$. Show that

$$V[0, t] - f(t) = V[0, s] - f(s) + f(s) - f(t) + V[s, t] \ge V[0, s] - f(s).$$

Hint: Consider a two-point grid on $[s, t]$.

(iv) Now suppose f is right-continuous at some $a \in \mathbb{R}^+$. For a fixed $b > a$ and an $\epsilon > 0$ choose a grid

$$\mathbb{G}: \quad a = t_0 < t_1 < \ldots < t_N = b$$

for which $V(\mathbb{G}, [a, b]) > V[a, b] - \epsilon$. With no loss of generality suppose $|f(t_1) - f(a)| < \epsilon$. Show that

$$\epsilon + V[t_1, b] \ge V(\mathbb{G}, [a, b]) > V[a, t_1] + V[t_1, b] - \epsilon$$

Deduce that $t \mapsto V[0, t]$ is continuous from the right at a.

(v) If f is right-continuous, show that $V_f[a, b]$ can be determined by taking a supremum over equispaced grids on $[a, b]$.

(vi) If X is an R-processes with sample paths of finite variation, show that it can be expressed as the difference of two R-processes with increasing sample paths. [The issue is whether $V_{X(\cdot,\omega)}[0, t]$ is adapted.]

[3] Suppose σ and τ are stopping times and $X \in \mathcal{SMG}$. With Y an $\mathcal{F}_\sigma$-measurable random variable, define $H = Y(\omega)((\sigma, \tau]]$. Show that $H \bullet X_t = Y(\omega)\left(X_{t\wedge\tau} - X_{t\wedge\sigma}\right)$ by the following steps.

(i) Start with the case where $Y = F \in \mathcal{F}_\sigma$. Define new stopping times $\sigma' = \sigma F + \infty F^c$ and $\tau' := \tau F + \infty F^c$. Show that

$$(F((\sigma, \tau]]) \bullet X_t = X_{t\wedge\tau'} - X_{t\wedge\sigma'} = F\left(X_{t\wedge\tau} - X_{t\wedge\sigma}\right).$$

(ii) Extend the equality to all bounded, $\mathcal{F}_\sigma$-measurable Y by a generating class argument.

(iii) For unbounded Y, define $H_n := Y\{|Y| \le n\}((\sigma, \tau]]$. Show that the sequence $\{H_n\}$ is locally uniformly bounded and it converges pointwise to H.

(iv) Complete the argument.

[4] If H and K are in $\text{loc}\mathcal{H}_{\text{Bdd}}$, and X is a semimartingale, show that $K \bullet (H \bullet X) = (KH) \bullet X$ for almost all paths. Hint: For fixed H, define $\psi(K) := (HK) \bullet M$. What do you get when $K = ((0, \tau]]$?

[5] Suppose $H, K \in \text{loc}\mathcal{H}_{\text{Bdd}}$ and $X, Y \in \mathcal{SMG}_0$. Show that $[H \bullet X, K \bullet Y] = (HK) \bullet [X, Y]$ by the following steps.

(i) Consider first the case where $K \equiv 1$. Show that $H \mapsto [H \bullet X, Y]$ and $H \mapsto H \bullet [X, Y]$ are both linear maps from $\text{loc}\mathcal{H}_{\text{Bdd}}$ into $\mathcal{SMG}$, which agree when $H = ((0, \tau]]$.

(ii) Use a λ-space argument followed by a localization to extend the result to $\text{loc}\mathcal{H}_{\text{Bdd}}$.

(iii) Invoke part (ii)—or trivial rearrangements thereof—twice to transform to an iterated stochastic integral.

$$[H \bullet X, K \bullet Y] = H \bullet [X, K \bullet Y] = H \bullet \big(K \bullet [X, Y]\big).$$

□ (iv) Invoke Problem [4] to complete the argument.

[6] Suppose $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$.

(i) Show that the process $X_t := M_t^2 - [M, M]_t$ belongs to $\text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$.

(ii) Suppose M has continuous sample paths and $[M, M]_t \equiv t$. Show that M is a standard Brownian motion.

References

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