PROJECT 8

$\mathcal{SMG}$ is nonstandard notation

An R-process X is called a ***semimartingale***, for a given standard filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$, if it can be decomposed as $X_t = X_0 + M_t + A_t$ with $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ and $A \in \mathcal{FV}_0$. Write $\mathcal{SMG}$ for the class of all semimartingales and $\mathcal{SMG}_0$ for those semimartinagles with $X_0 \equiv 0$.

1. Corrections

In my enthusiasm for a single definition of localization, which could be applied to both $\mathcal{M}_0^2(\mathbb{R}^+)$ and $\mathcal{H}_{\text{Bdd}}$, I created an awkward problem for processes H defined only on $(0,\infty)\times\Omega$. If $H_0(\omega)$ is not defined, what does $H(t\wedge\tau(\omega),\omega)$ mean at those ω for which $\tau(\omega)=0$? It would be much better to follow traditional and defineloc$\mathcal{H}_{\text{Bdd}}$ to consist of those predictable processes H for which there exist stopping times $\tau_k \uparrow \infty$ and finite constants C_k such that

$$|H((0,\tau_k]]| \le C_k \qquad \text{for each } k.$$

Notice that there are no longer problems at ω for which $\tau_k(\omega)=0$, because $\{t\in\mathbb{R}^+ : 0 < t \le \tau_k(\omega)\} = \emptyset$ for such ω.

Similarly, I should have defined local uniform boundedness of a sequence $\{H_n\}$ in loc$\mathcal{H}_{\text{Bdd}}$ to mean existence of stopping times $\tau_k \uparrow \infty$ and finite constants C_k such that

$$|H_n((0,\tau_k]]| \le C_k \qquad \text{for each } n \text{ and } k.$$

I was also too vague about the definition of L-processes on $\mathbb{R}^+\times\Omega$. Should such a process X be defined at $t=0$? Should we require existence of a finite right-hand limit at $t=0$? Should we require existence of a limit as $t\to\infty$? To make sense of my assertion that L-processes belong to loc$\mathcal{H}_{\text{Bdd}}$, I should regard X as an adapted process defined on $(0,\infty)\times\Omega$ with sample paths that are left-continuous on $(0,\infty)$, with no assumptions about the behavior as $t\to\infty$. I also need existence of a finite right limit at each t in $[0,\infty)$. With these assumptions, the stopping times

$$\tau_k := \inf\{t\in\mathbb{R}^+ : |X_t| > k\}$$

have the property that $|X((0,\tau_k]]| \le k$. Also we have $\tau_k\uparrow\infty$, because $X(\cdot,\omega)$ is bounded on each bounded interval $(0,M]$: You need a compactness argument to get a covering of $[0,M]$ by finitely many intervals $(t_i-\delta_i, t_i+\delta_i)$ within which

$$\begin{aligned} |X(t,\omega) - X(t_i,\omega)| &\le \epsilon \qquad \text{for } t_i-\delta_i < t \le t_i \\ |X(t,\omega) - X(t_i+,\omega)| &\le \epsilon \qquad \text{for } t_i < t \le t_i+\delta_i. \end{aligned}$$

I have also decided that it would be better to slightly change parts (iii) and (vi) of the basic theorem about semimartinagles, to simplify one step in the typical generating class argument.

linear in almost sure sense

<1> **Theorem.** *For each X in $\mathcal{SMG}_0$, there is a linear map $H \mapsto H \bullet X$ from* $\text{loc}\mathcal{H}_{\text{Bdd}}$ *into $\mathcal{SMG}_0$ such that:*

(i) $((0,\tau]] \bullet X_t = X_{t\wedge\tau}$ *for each $\tau \in \mathcal{T}$ and $t \in \mathbb{R}^+$.*

(ii) $H \bullet X_{t\wedge\tau} = (H((0,\tau]]) \bullet X_t = H \bullet (X_{\wedge\tau})_t$ *for each $\tau \in \mathcal{T}$ and $t \in \mathbb{R}^+$.*

(iii) If a sequence $\{H^{(n)} : n \in \mathbb{N}\} \subseteq \text{loc}\mathcal{H}_{\text{Bdd}}$ is locally uniformly bounded and $H^{(n)}(t,\omega) \to H(t,\omega)$ for each (t,ω), then $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$ and $H^{(n)} \bullet X \xrightarrow{ucpc} H \bullet X$.

Conversely, let ψ be another linear map from $\text{loc}\mathcal{H}_{\text{Bdd}}$ into $\mathcal{SMG}_0$ having at least the weaker properties:

(iv) $\psi(((0,\tau]])_t = X_{t\wedge\tau}$ almost surely, for each $\tau \in \mathcal{T}$ and $t \in \mathbb{R}^+$.

(vi) If a sequence $\{H^{(n)} : n \in \mathbb{N}\} \subseteq \text{loc}\mathcal{H}_{\text{Bdd}}$ is locally uniformly bounded and $H^{(n)}(t,\omega) \to H(t,\omega)$ for each (t,ω), then $\psi(H^{(n)})_t \to \psi(H)_t$ in probability, for each fixed t.

Then $\psi(H)_t = H \bullet X_t$ almost surely for every t.

REMARK. I did attempt to weaken the pointwise convergence assumptions in (iii) and (vi) to: $H_t^{(n)} \to H_t$ almost surely for each t. Unfortunately, this change complicates (invalidates?) the argument that $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$. I do not know whether it is worthwhile attempting such a modification.

2. Quadratic variation

From Project 7:

<2> **Definition.** *The quadratic variation process of an X in $\mathcal{SMG}_0$ is defined as $[X,X]_t := X_t^2 - 2(X^{\ominus} \bullet X)_t$ for $t \in \mathbb{R}^+$. For general $Z \in \mathcal{SMG}$, define $Z, Z] := [X,X]$ where $X_t := Z_t - Z_0$.*

<3> **Definition.** *A **random grid** $\mathbb{G}$ is defined by a finite sequence of finite stopping times $0 \le \tau_0 \le \tau_1 \le \ldots \le \tau_k$. The mesh of the grid is defined as* $\text{mesh}(\mathbb{G}) := \max_i |\tau_{i+1} - \tau_i|$*; the max of the grid is defined as* $\max(\mathbb{G}) := \tau_k$.

<4> **Theorem.** *Suppose $X \in \mathcal{SMG}$ and $\{\mathbb{G}_n\}$ is a sequence of random grids with* $\text{mesh}(\mathbb{G}_n) \xrightarrow{a.s.} 0$ *and* $\max(\mathbb{G}_n) \xrightarrow{a.s.} \infty$*. Then:*

(i) $\sum_{\mathbb{G}_n} \left(X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}\right)^2 \xrightarrow{ucpc} [X,X]_t$.

(ii) The process $[X,X]$ has increasing sample paths;

Mention jumps as well?

(iii) If τ is a stopping time then $[X_{\wedge\tau}, X_{\wedge\tau}] = [X,X]_{\wedge\tau}$.

REMARK. It would perhaps be cleaner to assume $\text{mesh}(\mathbb{G}_n) \to 0$ and $\max(\mathbb{G}_n) \to \infty$ for every ω, to fit with the pointwise convergence assumptions in Theorem <1>. This effect could also be achieved by changing each τ_k on a negligible set. For a standard filtration, the change could be made without disturbing any measurability assumptions.

The ***square bracket process*** $[X,Y]$ of two semimartingales X and Y (also known as the quadratic covariation process of X and Y) is defined, by polarization, as

$$4[X,Y] := [X+Y, X+Y] - [X-Y, X-Y].$$

If $X_0 \equiv 0$ and $Y_0 \equiv 0$ then $4[X,Y]_t$ equals

$$(X_t+Y_t)^2 - (X_t-Y_t)^2 - 2(X+Y)^{\ominus} \bullet (X+Y)_t + 2(X-Y)^{\ominus} \bullet (X-Y)_t$$

<5> $$= 4X_tY_t - 4X^{\ominus} \bullet Y_t - 4Y^{\ominus} \bullet X_t.$$

Notice that $[X, Y]$ is equal to the quadratic variation process $[X, X]$ when $X \equiv Y$. Notice also that $[X, Y] \in \mathcal{FV}_0$, being a difference of two increasing processes started at 0.

The square bracket process inherits many properties from the quadratic variation. For example, a polarization argument derives the following result from Theorem <4>.

<6> **Theorem.** *Suppose $X, Y \in \mathcal{SMG}$ and $\{\mathbb{G}_n\}$ is a sequence of random grids with $\text{mesh}(\mathbb{G}_n) \xrightarrow{a.s.} 0$ and $\max(\mathbb{G}_n) \xrightarrow{a.s.} \infty$. Then*

<7>
$$\sum\nolimits_{\mathbb{G}_n} \left(X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}\right)\left(Y_{t\wedge\tau_{i+1}} - Y_{t\wedge\tau_i}\right) \xrightarrow{ucpc} [X, Y]_t,$$

and $[X_{\wedge\tau}, Y_{\wedge\tau}] = [X_{\wedge\tau}, Y] = [X, Y_{\wedge\tau}] = [X, Y]_{\wedge\tau}$ for each stopping time τ,

I am slightly suspicious of the following argument.

Outline of last part of the proof. Temporarily write $W_n(t, X, Y)$ for the sum on the left-hand side of <7>.

- Show that there is no loss of generality in assuming that τ is one of the grid points. Hint: Consider a new grid

$$0 = \tau_0 \wedge \tau \le \tau_1 \wedge \tau \le \ldots \le \tau_k \wedge \tau \le \tau \le \tau \vee \tau_1 \le \ldots \le \tau \vee \tau_k$$

- For the grid $\mathbb{G}n$, suppose $\tau = \tau_\ell$. Show that

$$\begin{aligned} W_n(t \wedge \tau, X, Y) &= \sum\nolimits_{i=0}^{\ell-1} \left(X_{t\wedge\tau_{i+1}} - X_{t\wedge\tau_i}\right)\left(Y_{t\wedge\tau_{i+1}} - Y_{t\wedge\tau_i}\right) \\ &= W_n(t, X_{\wedge\tau}, Y) = W_n(t, X, Y_{\wedge\tau}) = W_n(t, X_{\wedge\tau}, Y_{\wedge\tau}) \end{aligned}$$

- Invoke uniform convergence over $[0, t]$ in probability. □

3. Itô formulae

Treat processes with jumps, or just cite Dellacherie & Meyer (1982, §VIII.24–28) or Protter (1990, page 71)?

Suppose X and Y are semimartingales with continuous paths, such that the two-dimensional random process $\{(X_t, Y_t) : t \in \mathbb{R}^+\}$ takes values in an open subset G of $\mathbb{R}^2$. Suppose Y has paths of bounded variation.

Let f be a continuous, real-valued function on G with two continuous partial derivatives f_x and f_{xx} with respect to its first argument and a continuous partial derivative f_y with respect to its second argument.

Define new processes by

$$\begin{aligned} F_x(s, \omega) &:= f_x\big(X(s, \omega), Y(s, \omega)\big), \\ F_{xx}(s, \omega) &:= f_{xx}\big(X(s, \omega), Y(s, \omega)\big) \\ F_y(s, \omega) &:= f_y\big(X(s, \omega), Y(s, \omega)\big). \end{aligned}$$

Each of them is adapted and has continuous paths; each process is predictable.

<8> **Itô Formula.** *The process $f(X_s, Y_s)$ is a semimartingale with*

$$f(X_t, Y_t) - f(X_0, Y_0) = (F_x \bullet X)_t + {}^1\!/\!_2 \left(F_{xx} \bullet [X, X]\right)_t + (F_y \bullet Y)_t$$

for each t in $\mathbb{R}^+$.

REMARK. The Itô formula is often written in the suggestive form

$$df(X_t, Y_t) = f_x(X_t, Y_t)\, dX_t + {}^1\!/\!_2 f_{xx}(X_t, Y_t)\, d[X, X]_t + f_y(X_t, Y_t)\, dY_t,$$

which hints at its origins as a sum of small increments.

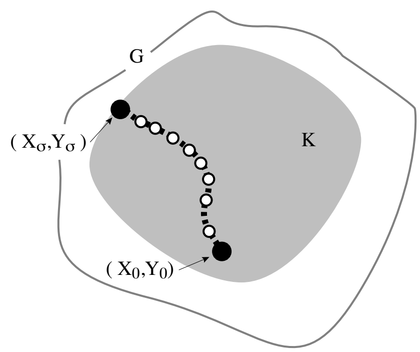
Proof. Let K be a compact subset of G. Define

$$\sigma := \inf\{t \in \mathbb{R}^+ : (X_t, Y_t) \notin K\}.$$

I was tired when sketching the proof. Beware of stupidities.

Replace X and Y by the corresponding stopped processes $X_{\wedge\sigma}$ and $Y_{\wedge\sigma}$.

- Show that the formula is trivially true for the stopped processes if $(X_0, Y_0) \notin K$.
- For each $\epsilon > 0$ show that there exists a $\delta > 0$ for which: if $(x, y) \in K$ and $\max(|\Delta x|, |\Delta y|) \leq \delta$ then
$$|f_\square(x + \Delta x, y + \Delta y) - f_\square(x, y)| \leq \epsilon \qquad \text{where } \square = x \text{ or } xx \text{ or } y.$$
- For $\max(|\Delta x|, |\Delta y|) \leq \delta$ and $(x, y) \in K$, show that
$$\begin{aligned} &f(x+\Delta x, y + \Delta y) - f(x, y) \\ &\qquad = (\Delta x) f_x(x, y) + 1/2(\Delta x)^2 f_{xx}(x, y) + (\Delta y) f_y(x, y) + \text{REM} \\ &\qquad\quad \text{where REM} \leq \epsilon\big(1/2(\Delta x)^2 + |\Delta y|\big) \end{aligned}$$
- Fix t. Let δ_n correspond to some sequence $\epsilon_n \downarrow 0$. Define a grid $\mathbb{G}_n$ via stopping times
$$\tau_{i+1} := \inf\{s \geq \tau_i : |(X, Y)_s - (X, Y)_{\tau_i}| \geq \delta_n\} \wedge t \wedge \sigma.$$
Show that there exist integers $k(n)$ such that $\mathbb{P}\{\tau_{k(n)} = t \wedge \sigma\} \to 1$ as $n \to \infty$.



- Write $\Delta_i X$ for $X_{\tau_{i+1}} - X_{\tau_i}$, and similarly for Y. Show that $f(X_{\tau_{k(n)}}, Y_{\tau_{k(n)}}) - f(X_0, Y_0)$ differs from

<9> $$\sum_{i=0}^{k(n)-1} (\Delta_i X) F_x(\tau_i) + 1/2(\Delta_i X)^2 F_{xx}(\tau_i) + (\Delta_i Y) F_y(\tau_i)$$

by a quantity that tends in probability to zero.
- Show that the contribution from the first summand in <9> equals $(H_n \bullet X)_t$, where
$$H_n(s, \omega) = \sum_{i=0}^{k(n)-1} F_x(\tau_i, \omega)((\tau_i.\tau_{i+1}]],$$
which is uniformly bounded and converges pointwise to F_x.
- Deduce that
$$\sum_{i=0}^{k(n)-1} (\Delta_i X) F_x(\tau_i) \xrightarrow{ucpc} F_x \bullet X_{t\wedge\sigma}$$
- Argue similarly for the contribution from the third summand in <9> .
- Define $Z_t := X_t - X_0$. Abbreviate $Z_{\tau_{i+1}} - Z_{\tau_i}$ to $\Delta_i Z$. Show that

cf. Protter (1990, page 69)

$$\begin{aligned} \sum_{i=0}^{k(n)-1} (\Delta_i X)^2 F_{xx}(\tau_i) &= \sum_{i=0}^{k(n)-1} F_{xx}(\tau_i)(Z^2_{\tau_{i+1}} - Z^2_{\tau_i}) \\ &\quad - 2\sum_{i=0}^{k(n)-1} \big(F_{xx}(\tau_i) Z_{\tau_i}\big)(\Delta_i Z). \end{aligned}$$
- Show that the right-hand side converges in probability to
$$\begin{aligned} F_{xx} \bullet Z^2_{t\wedge\sigma} - 2(F_{xx} Z) \bullet Z_{t\wedge\sigma} &= F_{xx} \bullet (Z^2 - 2Z \bullet Z)_{t\wedge\sigma} \\ &= F_{xx} \bullet [Z, Z]_{t\wedge\sigma} = F_{xx} \bullet [X, X]_{t\wedge\sigma}. \end{aligned}$$

- Deduce that

$$\begin{aligned} f(X_{t\wedge\sigma}, Y_{t\wedge\sigma}) &- f(X_0, Y_0) \\ &= (F_x \bullet X_{\wedge\sigma})_t + 1/2(F_{xx} \bullet [X_{\wedge\sigma}, X_{\wedge\sigma}])_t + (F_y \bullet Y_{\wedge\sigma})_t \\ &= (F_x \bullet X)_{t\wedge\sigma} + 1/2(F_{xx} \bullet [X, X])_{t\wedge\sigma} + (F_y \bullet Y)_{t\wedge\sigma}. \end{aligned}$$

□ • Complete the proof by letting K expand up to G, so that $\sigma \uparrow \infty$.

REMARKS.

(i) There would be nothing to gain by requiring existence of second-order partial derivatives f_{xy} and f_{yy}, because the corresponding bracket process $[X, Y]$ and $[Y, Y]$ are both zero—the process Y has paths of finite variation.

(ii) The process $1/2F_{xx} \bullet [X, X] + F_y \bullet Y$ is in $\mathcal{FV}$. If $X \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ then $F_x \bullet X \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$. The Itô formula then gives the semimartingale decomposition for the process $f(X_t, Y_t)$.

The story in Remark (i) changes if Y does not have paths of bounded variation. The error term $\epsilon_n \sum_i |\Delta_i Y|$ would no longer disappear in the limit. We would instead need continuous second order partial derivatives f_{xy} and f_{yy} to handle the contributions from the $\Delta_i Y$ increments to the Taylor expansion (to quadratic terms) in both variables. Error terms like

$$\epsilon_n \sum_i (\Delta_i Y)^2 + (\Delta_i X)(\Delta_i Y)$$

would again converge in probability to zero. The cross-product term

$$\begin{aligned} &\sum_i F_{xy}(\tau_i)(\Delta_i X)(\Delta_i Y) \\ &\quad = \sum_i F_{xy}(\tau_i)(X_{\tau_{i+1}} Y_{\tau_{i+1}} - X_{\tau_i} Y_{\tau_i}) - \sum_i F_{xy}(\tau_i)\left(X_{\tau_i}(\Delta_i Y) + Y_{\tau_i}(\Delta_i X)\right) \end{aligned}$$

would converge in probability to

$$F_{xy} \bullet (XY - X_0 Y_0 - X \bullet Y - Y \bullet X)_t = F_{xy} \bullet [X, Y]_t.$$

A similar argument works for functions of more than two semimartingales.

<10> **Multiprocess Itô Formula.** *Suppose $X^{(1)}, \dots X^{(d)}$ and $Y^{(1)}, \dots Y^{(d')}$ are semimartingales with continuous paths, such that the $d + d'$-dimensional random process $(\mathbf{X}, \mathbf{Y})$ takes values in an open subset G of $\mathbb{R}^{d+d'}$. Suppose each $Y^{(\gamma)}$ has paths of finite variation.*

If f is a continuous, real-valued function on G with continuous partial derivatives $f_{x(\alpha)}$, $f_{x(\alpha),x(\beta)}$, $f_{y(\gamma)}$ for $\alpha, \beta = 1, \dots, d$ and $\gamma = 1, \dots, d'$, then $f(\mathbf{X}, \mathbf{Y})$ is a semimartingale with

$$\begin{aligned} f(\mathbf{X}_t, \mathbf{Y}_t) - f(\mathbf{X}_0, \mathbf{Y}_0) &= \sum_\alpha F_{x(\alpha)} \bullet X_t^{(\alpha)} \\ &\quad + \sum_\gamma F_{y(\gamma)} \bullet Y_t^{(\gamma)} + 1/2 \sum_{\alpha,\beta} F_{x(\alpha),x(\beta)} \bullet [X^{(\alpha)}, X^{(\beta)}]_t \end{aligned}$$

for each t in $\mathbb{R}^+$.

<11> **Example.** Let $\{X_t : t \in \mathbb{R}^+\}$ be a locally square integrable martingale with continuous sample paths. Its quadratic variation process $Y := [X, X]$ is continuous (invoke the ucpc of the sum of squared increments) and of bounded variation. To be on the safe side, let me also assume that $X_0 \equiv 0$, even though it is not necessary.

Remove assumption on X_0.

The semimartingale $Z_t := \exp(X_t - 1/2Y_t)$ is a candidate for an application of the Itô formula, with $f(x, y) = \exp(x - 1/2y)$. We have $F_x = F_{xx} = -2F_y = Z$, and

$$Z_t - Z_0 = Z \bullet X + 1/2Z \bullet [X, X]_t - 1/2Z \bullet Y_t = Z \bullet X_t.$$

The Z process is also a locally square integrable martingale with continuous paths. □

4. Problems

[1] Show that

$$[X_1 + X_2, Y_1 + Y_2] = [X_1, Y_1] + [X_1, Y_2] + [X_2, Y_1] + [X_2, Y_2],$$

for semimartingales X_1, X_2, Y_1, and Y_2.

[2] Suppose $X \in \mathcal{SMG}$ and $Y \in \mathcal{FV}$. Suppose that X has continuous sample paths. Show that $[X, Y]_t = 0$ almost surely, for each t. Hint: Consider a random grid defined by

$$\tau_{i+1} := (\tau_i + n^{-1}) \wedge \min\{t \geq \tau_i : |X(t) - X(\tau_i)| \geq n^{-1}\}.$$

[3] For H_1, H_2, K_1, K_2 in loc$\mathcal{H}^\infty$, and X_1, X_2, Y_1, Y_2 in $\mathcal{SMG}$, show that

$$\begin{aligned}[t] &[H_1 \bullet X_1 + K_1 \bullet Y_1, H_2 \bullet X_2 + K_2 \bullet Y_2] \\ &\quad = (H_1 H_2) \bullet [X_1, X_2] + (H_1 K_2) \bullet [X_1, Y_2] \\ &\quad\quad + (K_1 H_2) \bullet [Y_1, X_2] + (K_1 K_2) \bullet [Y_1, Y_2]. \end{aligned}$$

[4] If $X \in \mathcal{SMG}$ has continuous paths, show that $[X, X]$ also has conntinuous paths.

References

Dellacherie, C. & Meyer, P. A. (1982), *Probabilities and Potential B: Theory of Martingales*, North-Holland, Amsterdam.

Protter, P. (1990), *Stochastic Integration and Differential Equations*, Springer, New York.