Project 9

Itô Formula: Two-dimensional semimartingale (X, Y) with continuous paths and $Y \in \mathcal{FV}$, which ensures $[Y, Y] = 0$. Continuous, real-valued function f with enough continuous partial derivatives to define predictable processes

$$F_x(s,\omega) := f_x\big(X(s,\omega), Y(s,\omega)\big),$$
$$F_{xx}(s,\omega) := f_{xx}\big(X(s,\omega), Y(s,\omega)\big)$$
$$F_y(s,\omega) := f_y\big(X(s,\omega), Y(s,\omega)\big).$$

with continuous sample paths. Then $f(X_s, Y_s)$ is a semimartingale with

$$f(X_t, Y_t) - f(X_0, Y_0) = (F_x \bullet X)_t + {}^1\!/\!_2 \big(F_{xx} \bullet [X, X]\big)_t + (F_y \bullet Y)_t$$

for each t in $\mathbb{R}^+$.

1. Corrections

On Project 7 I asked you to show that

<1> $$[H \bullet X, Y] = H \bullet [X, Y] \qquad \text{for } X, Y \in \mathcal{SMG}_0 \text{ and } H \in \text{loc}\mathcal{H}_{\text{Bdd}}.$$

I implied that the proof was just a simple example of a generating class argument. As some of you discovered, the proof is a little more delicate. A clean argument can be extracted from ideas used by Protter (1990, section II.6).

<2> **Lemma.** *Suppose $\{H_n : n \in \mathbb{N}\} \subseteq \text{loc}\mathcal{H}_{\text{Bdd}}$ is locally uniformly bounded and $H_n \xrightarrow{ucpc} 0$. Suppose also that $Y \in \mathcal{SMG}$. Then $H_n \bullet Y \xrightarrow{ucpc} 0$.*

Proof. Suppose there is a t for which $\sup_{s \le t} |H_n \bullet Y_s|$ does not converge to zero in probability. For some $\epsilon > 0$ there is a subsequence along which $\mathbb{P}\{\sup_{s\le t} |H_n \bullet Y_s| > \epsilon\} > \epsilon$. Along a subsubsequence we have the same inequality as well as $\sup_{s\le t} |H_n(s)| \to 0$ almost surely; along the subsubsequence $\sup_{s\le t} |\{\omega \in N^c\} H_n(s,\omega)| \to 0$ for every ω, for some negligible set N. The sequence $K_n := \{\omega \in N^c\} H_n((0, t]]$ is locally uniformly bounded and it converges pointwise to zero. Each K_n is $\mathcal{P}$-measurable, because $(0, 1] \times N \in \mathcal{P}$, and

$$\mathbb{P}\{\sup_{s\le t} |K_n \bullet Y_s - H_n \bullet Y_s| \ne 0\} = 0.$$

Along the subsubsequence, $K_n \bullet Y \xrightarrow{ucpc} 0$, which contradicts the property defining the first subsequence. □

To establish assertion <1>, consider the linear map

$$\begin{aligned}\psi(H) &:= [H \bullet X, Y] - H \bullet [X, Y] \\ &= (H \bullet X)Y - (H \bullet X)^{\ominus} \bullet Y - Y^{\ominus} \bullet (H \bullet X) \\ &\qquad - H \bullet \big(XY - X^{\ominus} \bullet Y - Y^{\ominus} \bullet X\big) \\ &= (H \bullet X)Y - (H \bullet X)^{\ominus} \bullet Y - H \bullet \big(XY - X^{\ominus} \bullet Y\big)\end{aligned}$$

because $Y^{\ominus} \bullet (H \bullet X) = (Y^{\ominus} H) \bullet X = H \bullet \big(Y^{\ominus} \bullet X\big)$.

You can check that $\psi((0, \tau]] = 0$ for $\tau \in \mathcal{T}$ and that $\psi(H_n - H) \xrightarrow{ucpc} 0$ if $\{H_n : n \in \mathbb{N}\}$ is locally uniformly bounded and $H_n \to H$ pointwise. I think the rest of the argument is routine.

Please inform me if you find more gaps in the proof.

2. Exponential martingales

Suppose $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ has continuous sample paths. For $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$, define

<3> $$Z_t = \exp\left(iH \bullet M_t + \tfrac{1}{2}H^2 \bullet [M, M]_t\right)$$

- Invoke the complex analog of the Itô formula (or apply the result to real and imaginary parts) to show that

$$\begin{aligned} Z_t - 1 &= iZ \bullet (H \bullet M)_t - \tfrac{1}{2}Z \bullet [H \bullet M, H \bullet M]_t + \tfrac{1}{2}Z \bullet \left(H^2 \bullet [M, M]\right)_t \\ &= i(ZH) \bullet M_t \end{aligned}$$

  You may use any of the properties established in the problems for Project 8.
- Deduce that $Z - 1 \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$.

3. Lévy again

Define $\mathcal{U}_t = t$. Suppose $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ with continuous sample paths and $M^2 - \mathcal{U} \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$. Show that M is a standard Brownian motion.

- Use Problem [2] to explain why $[M, M] = \mathcal{U}$.
- For fixed constants $0 = t_0 < t_1 < \ldots < t_\ell < \infty$ and real numbers $\{\theta_j\}$ define

$$H = \sum\nolimits_{j=0}^{\ell-1} \theta_j ((t_j, t_{j+1}]]$$

  Show that, for $t \geq t_\ell$,

$$\begin{aligned} H \bullet M_t &= \sum\nolimits_j \theta_j \Delta_j M \qquad \text{where } \Delta_j M := M(t_{j+1}) - M(t_j) \\ H^2 \bullet \mathcal{U}_t &= \sum\nolimits_j \theta_j^2 \delta_j \qquad \text{where } \delta_j := t_{j+1} - t_j \end{aligned}$$

- For Z as in <3> and H as above, show that there is a sequence of stopping times $\tau_k \uparrow \infty$ for which $\mathbb{P}Z_{t \wedge \tau_k} = 1$ for all k.
- Invoke Dominated Convergence to deduce that

$$\mathbb{P}\exp\left(i \sum\nolimits_j \theta_j \Delta_j M\right) = \exp\left(-\tfrac{1}{2}\sum\nolimits_j \theta_j^2 \delta_j)\right)$$

- Conclude that M is a standard Brownian motion.

4. Brownian filtrations

Let $\{B(t, \omega : 0 \leq t \leq 1\}$ be a Brownian motion with continuous sample paths on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The ***Brownian filtration*** on Ω is defined by $\mathcal{F}_t := \sigma(\{B_s : 0 \leq s \leq t\} \cup \mathcal{N})$, where $\mathcal{N}$ denotes the class of all $\mathbb{P}$-negligible sets. The submartingale B^2 has Doléans measure $\mu = \mathfrak{m} \otimes \mathbb{P}$.

<4> **Definition.** *A cadlag process $\{M_t : 0 \leq t \leq 1\}$ is said to be a local martingale if there exist stopping times $\tau_k \uparrow \infty$ for which each $M_{\wedge \tau_k}$ is a martingale.*

Maybe better to restrict definition to cases where $M_0 = 0$.

Local martingales (with $M_0 = 0$) with respect to the Brownian filtration have two striking properties:

(i) They have continuous sample paths. Thus they all belong to $\text{loc}\mathcal{M}_0^2[0, 1]$.

(ii) They can be represented as stochastic integrals.

See Problem [4] for the first assertion. The second will follow via an argument based on the Itô formula.

<5> **Theorem.** *For each* $X \in \mathcal{L}^2(\Omega, \mathcal{F}_1, \mathbb{P})$ *there exists an* $H \in \mathcal{L}^2(\mu)$ *such that* $X - \mathbb{P}X = H \bullet B_1$ *almost surely.*

Sketch of proof. Without loss of generality, suppose $\mathbb{P}X = 0$.

- Show that $\mathcal{R} := \{H \bullet B_1 : H \in L^2(\mu)\}$ is a closed vector subspace of $\mathcal{L}^2(\mathbb{P}, \mathcal{F}_1)$. Hint: If $H_n \bullet B_1 \to Y$ in $\mathcal{L}^2(\mathbb{P})$-norm, show that $\{H_n\}$ is a Cauchy sequence with a limit H in $\mathcal{L}^2(\mu_1)$. Deduce that $Y = H \bullet B_1$.
- Let Z denote the component of X that is orthogonal to $\mathcal{R}$. That is, $X = Z + K \bullet B_1$ for some $K \in \mathcal{L}^2(\mu)$ and $\mathbb{P}Z(H \bullet B)_1 = 0$ for all H in $\mathcal{L}^2(\mu)$. Show that $\mathbb{P}Z = 0$.
- Explain why we need to prove $Z = 0$ almost surely.
- Explain why it suffices to show $\mathbb{P}Zf(B) = 0$ for all bounded, $\mathcal{C}$-measurable functionals f on $C[0, 1]$.
- Explain why it suffices to consider functionals f that depend on B only through its values at a finite set of times.
- Explain why it suffices to consider functionals f that depend on B only through its increments $Y_j = B_{t_{j+1}} - B_{t_j}$ for a fixed set of times $0 = t_0 < t_1 < \ldots < t_k = 1$. That is, why is it enough to prove $\mathbb{P}Zg(\mathbf{Y}) = 0$ for all bounded, measurable functions g on $\mathbb{R}^k$?
- Invoke Problem [3] to show that it is enough to prove $\mathbb{P}Z\exp(i\boldsymbol{\theta} \cdot \mathbf{Y}) = 0$ for all $\boldsymbol{\theta}$ in $\mathbb{R}^k$.
- Work with stochastic integral notation. Show that $\boldsymbol{\theta} \cdot \mathbf{Y} = H \bullet B_1$, where $H := \sum_{j=0}^{k-1} \theta_j((t_j, t_{j+1}]]$.
- Show that $H \bullet B$ has a deterministic quadratic variation process, $A_t := [H \bullet B, H \bullet B]_t = \int_0^t H^2(s)\,ds$.
- Use the results from Section 2 to show that
$$W_1 = 1 + i(WH) \bullet B_1 \qquad \text{where } W_t := \exp(iH \bullet B_t + {}^1\!/\!{}_2 A_t).$$
- Deduce that
$$\exp(A_1/2)\mathbb{P}Z\exp(i\boldsymbol{\theta} \cdot \mathbf{Y}) = 0.$$
- Are we done? □

<6> **Corollary.** *For each local martingale* M *adapted to the Brownian filtration there exists an* H *in* $\text{loc}\mathcal{L}^2(\mu)$ *such that* $M_t = M_0 + (H \bullet B)_t$ *for* $0 \le t \le 1$.

Proof. Without loss of generality, suppose $M_0 = 0$. Define stopping times $\tau_k := 1 \wedge \inf\{t : |M_t| \ge k\}$.

- Why does $M_{\wedge \tau_k}$ belong to $\mathcal{M}_0^2[0, 1]$?
- For each k, explain why there exists an $H_k \in \mathcal{L}_2(\mu)$ such that
$$M_{t \wedge \tau_k} = \big(H_k((0, \tau_k]]\big) \bullet B_t \qquad \text{for } 0 \le t \le 1.$$
- Deduce that $(H_k((0, \tau_k]]) \bullet B_1 = (H_{k+1}((0, \tau_k]] \bullet B_1$ almost surely.
- Deduce that $H_k((0, \tau_k]] - H_{k+1}((0, \tau_k]] = 0$ almost everywhere $[\mu]$.
- Show that the H_k processes can be pasted together to create an H in $\text{loc}\mathcal{L}^2(\mu)$ for which $M_t = H \bullet B_t$ almost surely. □

Am I just repeateing the construction for the $\text{loc}\mathcal{M}_0^2[0, 1]$ stochastic integral?

Remark. Should I extend to general $\mathcal{F}_1$-measurable random variables, perhaps using the method of Dudley (1977), getting a representation $Y_0 + H \bullet B_1$ with $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$.

5. Problems

[1] Suppose $Z \in \mathcal{FV}_0 \cap \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ and Z has continuous sample paths. Show that $Z_t = 0$ almost surely, for each t. Hint: Use the fact that $[Z, Z] = 0$ to deduce that $Z^2 = 2Z \bullet Z \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$. Find a sequence of stopping times $\tau_k \uparrow \infty$ for which $\mathbb{P}Z_{t\wedge\tau_k}^2 = 0$ for each t.

[2] Suppose $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ has continuous sample paths. Suppose $A \in \mathcal{FV}_0$ also has continuous paths and $M^2 - A \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$. Deduce that $A = [M, M]$. Hint: Apply Problem [1] to $[M, M] - A$.

[3] Let X be an integrable random variable, and $\mathbf{Y} = (Y_1, \ldots, Y_k)$ be a vector of random variables such that $\mathbb{P}X \exp(i\boldsymbol{\theta} \cdot \mathbf{Y}) = 0$ for all $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ in $\mathbb{R}^k$. Show that $\mathbb{P}(Xg(\mathbf{Y})) = 0$ for all bounded, measurable g. Hint: Let $\mu^\pm$ be the measures with densities $X^\pm$ with respect to $\mathbb{P}$. Show that $\mathbf{Y}$ has the same Fourier transform, and hence the same distribution, under both μ^+ and μ^-. That is, $\mu^+ g(\mathbf{Y}) = \mu^- g(\mathbf{Y})$.

[4] Suppose $\{X_t : 0 \le t \le 1\}$ is a cadlag martingale with respect to the Brownian filtration. Remember that X_1 can be expressed as $f(B)$ for some $\mathcal{C}\backslash\mathcal{B}(\mathbb{R})$-measurable functional f on $C[0, 1]$. The functional is $\mathbb{W}$-integrable.

(i) If f is a continuous (for sup-norm distance) functional on $C[0, 1]$, use the representation $X_t = \mathbb{P}_t f(B) = \mathbb{W}^x f(K_t B + S_t x)$ almost surely to show that X has continuous sample paths (almost surely?).

(ii) For a general $\mathbb{W}$-integrable functional, show that there exists a sequence of continuous functionals $\{f_n\}$ for which $\mathbb{W}|f - f_n| \le 4^{-n}$.

(iii) Let M_n be a version of the martingale $\mathbb{P}_t f_n(B)$ with continuous sample paths. Show that $|M_n(t) - X(t)|$ is a uniformly integrable submartingale with cadlag sample paths.

(iv) Define stopping times $\tau_n := 1 \wedge \min\{t : |M_n(t) - X(t)| \ge 2^{-n}\}$. Show that

$$
\begin{aligned}
\mathbb{P}\{\sup_t |M_n(t) - X(t)| > 2^{-n}\} &\le 2^n \mathbb{P}|M_n(\tau_n) - X(\tau_n)| \\
&\le 2^n \mathbb{P}|M_n(1) - X(1)| = 2^n \mathbb{P}|f_n(B) - f(B)| \\
&\le 2^{-n}
\end{aligned}
$$

(v) Deduce that $\sum_n \mathbb{P}\{\sup_t |M_n(t) - X(t)| > 2^{-n}\} < \infty$ and hence $\sup_t |M_n(t) - X(t)| \to 0$ almost surely.

(vi) Conclude that almost all sample paths of X are continuous.

(vii) Extend the argument to the case of a local martingale. Hint: If $M_{\wedge\tau_k}$ has (almost all) continuous paths for each k, and if $\tau_k \uparrow \infty$, what do you know about almost all paths of M?

[5] Let X and Y be independent Brownian Motions.

(i) Show that both $(X + Y)/\sqrt{2}$ and $(X - Y)/\sqrt{2}$ are also Brownian Motions.

(ii) Deduce that $[X, Y] = 0$.

The next problem presents the standard example of a uniformly integrable local martingale that is not of class [D].

Better to start at origin, and work with distance to $\mathbf{u}$?

[6] Let $\mathbf{B} = (1 + X, Y, Z)$ be a three-dimensional Brownian Motion started from $\mathbf{u} = (1, 0, 0)$. (The three processes X, Y, and Z are independent Brownian Motions started from zero.) Write $\mathbf{x}$ for (x, y, z). Define $f(\mathbf{x}) = 1/\|\mathbf{x}\|$ on $\mathbb{R}^3\backslash\{0\}$. Define a process $M(t) = 1/\|\mathbf{B}(t)\|$.

(i) Use the Multiprocess Itô Formula to show that $M \in \text{loc}\mathcal{M}^2(\mathbb{R}^+)$. Hint: Show that on the open region $\mathbb{R}^3\backslash\{0\}$ the function f is harmonic:

$$\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z} = 0.$$

(ii) Deduce that M is a positive supermartingale.

(iii) Let $\tau_k = \inf\{t : \|\mathbf{B}(t)\| \le 1/k\}$. Show that $M_{\wedge\tau_k} \in \mathcal{M}^2(\mathbb{R}^+)$.

(iv) Show that $C_0 := \int\{\|\mathbf{x}\| \le 1/2\}\|\mathbf{x}\|^{-2}\, d\mathbf{x} < \infty$.

(v) Show that $\mathbb{P}M(t)^2 \le C_0 \exp(-(8t)^{-1})t^{-3/2} + \mathbb{P}\left(8 \wedge \|\mathbf{B}(t)\|^{-2}\right)$.

(vi) Show that $\|\mathbf{B}(t)\|^2 \xrightarrow{\mathbb{P}} \infty$ as $t \to \infty$.

(vii) Deduce that $\sup_t \mathbb{P}M(t)^2 < \infty$ and $\mathbb{P}M(t) \to 0$ as $t \to \infty$.

(viii) Deduce that M is not a martingale, and hence M is not in class [D].

References

Dudley, R. M. (1977), 'Wiener functionals as Itô integrals', *Annals of Probability* **5**, 140–141.

Protter, P. (1990), *Stochastic Integration and Differential Equations*, Springer, New York.