Appendix C

Doléans measures

C.1 Introduction

Once again all random processes will live on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t : 0 \le t \le 1\}$. We should probably assume the filtration is standard even though it does not seem essential for this project.

The construction (Project 6) of the isometric stochastic integral $H \bullet M$ with respect to a martingale $M \in \mathcal{M}_0^2[0,1]$, at least for bounded, predictable H, depended on the existence of the Doléans measure μ on the predictable sigma-field $\mathcal{P}$ on $\mathfrak{S}_1 := \Omega \times (0,1]$. To make the map $H \mapsto H \bullet M_1$ an isometry between $\mathcal{H}_{\text{simple}}$ and a subset of $\mathcal{L}^2(\Omega, \mathcal{F}_1, \mathbb{P})$ we needed

<1> $$\mu(a,b] \times F = \mathbb{P}\{\omega \in F\}(M_b - M_a)^2 \qquad \text{for all } 0 \le a < b \le 1 \text{ and } F \in \mathcal{F}_a.$$

This property characterizes the measure μ because the collection of all predictable sets of the form $(a,b] \times F$ is $\cap f$-stable and it generates $\mathcal{P}$.

The sigma-field $\mathcal{P}$ is also generated by the set of all stochastic intervals $((0,\tau]]$ for $\tau \in \mathcal{T}_1$, the set of all $[0,1]$-valued stopping times for the filtration. The Doléans measure is also characterized by the property

$$\mu((0,\tau]] = \mathbb{P}M_\tau^2 \qquad \text{for all } \tau \in \mathcal{T}_1.$$

Notice that μ depends on M only through the submartingale $S_t := M_t^2$. In fact, analogous measures can be defined for a large class of submartingales.

<2> **Definition.** *Let $\{S_t : 0 \le t \le 1\}$ be a cadlag submartingale with $S_0 \equiv 0$. Say that a finite (countably-additive) measure μ_S, defined on the predictable sigma-field $\mathcal{P}$ on $\mathfrak{S}$, is the* ***Doléans measure*** *for S if $\mu((0,\tau]] = \mathbb{P}S_\tau$ for every τ in $\mathcal{T}_1$, the set of all $[0,1]$-valued stopping times for the filtration.*

Remark. If μ_S exists then S_τ must be integrable for every τ in $\mathcal{T}_1$. In fact we will need the uniform integrability only for stopping times with a finite range (Métivier 1982, page 80).

I mentioned explicitly that μ_S must be countably-additive to draw attention to a subtle requirement on S for μ_S to exist, known somewhat cryptically as ***property [D]***:

$[D]$ the set of random variables $\{S_\tau : \tau \in \mathcal{T}_1\}$ is uniformly integrable.

Problem [2] shows that if there exists a (countably-additive) Doléans measure μ_S then S has property [D]. The proof of the converse assertion is the main subject of this Appendix.

<3> **Theorem.** *Every cadlag submartingale $\{S_t : 0 \le t \le 1\}$ with property [D] has a countably additive Doléans measure μ_S on the predictable sigma-field $\mathcal{P}$ on $\mathfrak{S} := \Omega \times (0,1]$.*

Remark. Alert readers will be thinking—correctly—that property [D] must be used somewhere to convert a pointwise convergence along a sequence of stopping times into an $\mathcal{L}^1$ convergence.

<4> **Example.** If $M \in \mathcal{M}_0^2[0,1]$ then the submartingale $S_t = M_t^2$ has property $[D]$. Indeed, we know from Project 4 that $M_\tau = \mathbb{P}(M_1 \mid \mathcal{F}_\tau)$ for all $\tau \in \mathcal{T}_1$. Thus $0 \le M_\tau^2 \le \mathbb{P}\left(M_1^2 \mid \mathcal{F}_\tau\right)$ for each τ in $\mathcal{T}_1$. Use the fact that $\{\mathbb{P}(\xi \mid \mathcal{G}) : \mathcal{G}$ a sub-sigma-field of $\mathcal{F}\}$ is uniformly integrable for each integrable random variable ξ to complete the argument.

□

Remark. If I write μ_S for the Dol'eans measure then I should write μ_{M^2} instead of the μ_M used in Project 6. Indeed, Definition 2 would give $\mu_M \equiv 0$ if applied directly to the submartingale M. You might want to go back and make the necessary changes.

<5> **Example.** Let $\{B_t : 0 \le t \le 1\}$ be a standard Brownian motion. The submartingale $S_t := B_t^2$ has a very simple Doléans measure, characterized by

$$\mu_S(a,b] \otimes F = \mathbb{P}F\left(B_b^2 - B_a^2\right) = \mathbb{P}F(b-a) \qquad \text{for } F \in \mathcal{F}_a.$$

That is, $\mu_S = \mathbb{P} \otimes \mathfrak{m}$, with $\mathfrak{m}$ equal to Lebesgue measure on $\mathcal{B}(0,1]$. Of course μ_S has a further extension to the product sigma-field $\mathcal{F} \otimes \mathcal{B}(0,1]$.

A Poisson process $\{N_t : 0 \le t \le 1\}$ with intensity 1 shares with Brownian motion the independent increment property. It is a submartingale with sample paths that are constant, except for jumps of size 1 corresponding to points of the process; the increment $N_t - N_s$ has a Poisson$(t-s)$ distribution. It has Doléans measure $\mu_N = \mathbb{P} \otimes \mathfrak{m}$, the same as the square of Brownian motion.

Clearly the Doléans measure does not uniquely determine the submartingale. But the only square integrable martingale M with continuous sample paths and Doléans measure $\mu_{M^2} = \mathbb{P} \otimes \mathfrak{m}$ is Brownian motion: if $F \in \mathcal{F}_s$ and $s < t$ then $\mathbb{P}F(M_t^2 - M_s^2) = \mu_{M^2} F \times (s,t] = (t-s)\mathbb{P}F$, from which it follows that $M_t^2 - t$ is a martingale with respect to $\{\mathcal{F}_t\}$. It follows from Lévy's characterization that M is a Brownian motion.

□

C.2 Existence of the Doléans measure for class [D]

There are several ways to prove that a cadlag submartingale $\{S_t : 0 \le t \le 1\}$ of class [D] has a countably additive Doléans measure. For example:

(i) Invoke an approximation by compact sets (see Métivier 1982, Chapter 3 or Chung and Williams 1990, Section 2.8, for example).

(ii) For the square of a continuous $\mathcal{M}_0^2[0,1]$-martingale M, prove directly the existence of an increasing process A (the quadratic variation process) for which $M_t^2 - A_t$ is a martingale (Chung and Williams 1990, Section 4.4).

(iii) Do something very general, as in Dellacherie and Meyer (1982, §7.1).

I will present a different method, based on the identification of measures on $\mathcal{P}$ with a certain kind of linear functional defined on the vector space $\mathcal{H}_{\text{BddLip}}$ of all adapted, continuous processes H on $(0,1] \times \Omega$ for which there exists a finite constant C_H such that

(i) $|H(t,\omega)| \le C_H$ for all (t,ω).

(ii) $|H(s,\omega) - H(t,\omega)| \le C_H|t-s|$ for all s, t, and ω.

The limit $H(0,\omega) := \lim_{t\downarrow\downarrow 0} H(t,\omega)$ exists for each ω. Each member of $\mathcal{H}_{\text{BddLip}}$ is predictable. Also, as you saw in Project 6, $\mathcal{H}_{\text{BddLip}}$ generates the sigma-field $\mathcal{P}$.

<*6*> **Theorem.** *There exists a one-to-one correspondence between finite measures on $\mathcal{P}$ and increasing linear functionals $\mu : \mathcal{H}_{\text{BddLip}} \to \mathbb{R}$ for which*

$$\mu(h_k) \downarrow 0 \qquad \textit{for each } \{h_k : nk \in \mathbb{N}\} \subseteq \mathcal{H}_{\text{BddLip}} \textit{ with } h_k \downarrow 0 \textit{ pointwise.}$$

See Pollard (2001, Appendix A) for a proof.

Proof (of Theorem 3) Construct μ as a limit of simpler functionals. For each n in $\mathbb{N}$ and $i = 0, 1, \dots, 2^n$ define $t_{i,n} := i/2^n$. Write $\mathbb{P}_{i,n}(\cdots)$ for expectations conditional on $\mathcal{F}(t_{i,n})$. Define $\Delta_{i,n} := S(t_{i+1,n}) - S(t_{i,n})$. Note that $\mathbb{P}_{i,n}\Delta_{i,n} \ge 0$ almost surely, by the submartingale property. For each H in $\mathcal{H}_{\text{BddLip}}$, define

$$\mu_n H := \sum\nolimits_{0\le i<2^n} \mathbb{P}\left(H(t_{i,n})\Delta_{i,n}\right) = \sum\nolimits_{0\le i<2^n} \mathbb{P}\left(H(t_{i,n})\mathbb{P}_{i,n}\Delta_{i,n}\right).$$

Clearly, μ_n is a linear functional on $\mathcal{H}_{\text{BddLip}}$. The second expression for $\mu_n H$ ensures that μ_n is increasing in H.

The proof involves four steps:

(a) Show that $\mu H := \lim_{n\to\infty} \mu_n H$ exists for each $H \in \mathcal{H}_{\text{BddLip}}$. Note that μ inherits its linearity and the increasing property from μ_n.

(b) For each nonnegative $H \in \mathcal{H}_{\text{BddLip}}$ and each $\epsilon > 0$ define $\tau(H,\epsilon) := \inf\{t : H(t,\omega) \geq \epsilon\} \wedge 1$. Show that $\mu H \leq \epsilon \mathbb{P} S_1 + C_H \mathbb{P}\left(S_1 - S_{\tau(H,\epsilon)}\right)$.

(c) Show that $\mu(h_k) \downarrow 0$ if $h_k \downarrow 0$ pointwise.

(d) Show that $\mu((0,\tau]] = \mathbb{P} S_\tau$ for each $\tau \in \mathcal{T}_1$.

Proof of (a).

For each $H \in \mathcal{H}$, show that the sequence $\{\mu_n H : n \in \mathbb{N}\}$ is Cauchy. Fix n and m with $n < m$. Define $J_i := \{j : t_{i,n} \leq t_{j,m} < t_{i+1,n}\}$. Then

$$
\begin{aligned}
&|\left(\sum\nolimits_{j\in J_i} \mathbb{P} H(t_{j,m})\Delta_{j,m}\right) - \mathbb{P} H(t_{i,n})\Delta_{i,n}| \\
&\qquad = |\sum\nolimits_{j\in J_i} \mathbb{P}\left(H(t_{j,m}) - H(t_{i,n})\right)\Delta_{j,m}| \\
&\qquad \leq \sum\nolimits_{j\in J_i} \mathbb{P}\left(|H(t_{j,m}) - H(t_{i,n})|\mathbb{P}_{j,m}\Delta_{j,m}\right) \\
&\qquad \leq \sum\nolimits_{j\in J_i} C_H 2^{-n} \mathbb{P}\Delta_{j,m} \\
&\qquad = C_H 2^{-n} \sum\nolimits_{j\in J_i} \mathbb{P}\Delta_{i,n}
\end{aligned}
$$

Sum over i to deduce that $|\mu_m H - \mu_n H| \leq C_H 2^{-n} \mathbb{P} S_1$, which tends to zero as n tends to infinity.

Proof of (b).

Temporarily write τ_n for the discretized stopping time obtained by rounding $\tau(H,\epsilon)$ up to the next integer multiple of 2^{-n}. Then

$$
\begin{aligned}
\mu_n H &\leq \sum\nolimits_{0\leq i<2^n} \mathbb{P}\left(\epsilon\{t_{i,n} < \tau_n\} + C_H\{t_{i,n} \geq \tau_n\}\right)\mathbb{P}_{i,n}\Delta_{i,n} \\
&\leq \epsilon \sum\nolimits_{0\leq i<2^n} \mathbb{P}\Delta_{i,n} + C_H \sum\nolimits_{0\leq i<2^n} \mathbb{P}\{t_{i,n} \geq \tau_n\}\Delta_{i,n} \\
&\leq \epsilon \mathbb{P} S_1 + C_H \mathbb{P}\left(S_1 - S_{\tau_n}\right).
\end{aligned}
$$

Let n tend to infinity. Uniform integrability of the sequence $\{S_{\tau_n}\}$ together with right-continuity of the sample paths of S gives the asserted inequality.

Proof of (c).

For a fixed $\epsilon > 0$, temporarily write σ_k for $\tau(H_k, \epsilon)$. By compactness of $[0,1]$, the pointwise convergence of the continuous functions, $H_k(\cdot, \omega) \downarrow 0$, is actually uniform. For each ω, the sequence $\{\sigma_k(\omega)\}$ not only increases to 1, it actually achieves the value 1 at some finite k (depending on ω). Uniform integrability of $\{S_{\sigma_k} : k \in \mathbb{N}\}$ and the analog of (b) for each H_k then give

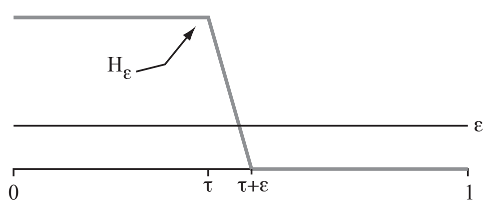
$$\mu H_k \leq \epsilon \mathbb{P} S_1 + C_H \mathbb{P}\left(S_1 - S_{\sigma_k}\right) \to \epsilon \mathbb{P} S_1 \qquad \text{as } k \to \infty.$$

By Theorem 6, the functional corresponds to the integral with respect to a finite measure on $\mathcal{P}$, with total mass $\mu((0,1]] = \lim_n \mu_n((0,1]] = \mathbb{P}S_1$.

Proof of (d).

For a given $\epsilon > 0$, approximate $((0, \tau]]$ by

$$H_\epsilon(t, \omega) := \min\left(1, (\tau(\omega) + \epsilon - t)^+ / \epsilon\right) \qquad \text{for } 0 \leq t \leq 1.$$



Note that $\{(\omega, t) \in \mathfrak{S} : H_\epsilon(\omega, t) \geq 1 - c\} = ((0, \tau + c\epsilon]]$ for each $0 \leq c < 1$, which ensures that H_ϵ is predictable. It belongs to $\mathcal{H}_{\text{BddLip}}$ with $C_{H_\epsilon} = 1/\epsilon$, and $\mu((0,\tau]] \leq \mu H_\epsilon \leq \mu((0, \tau + \epsilon]]$. When $2^{-n} < \epsilon$,

$$\begin{aligned}
\mathbb{P} S_{\tau + 2\epsilon} &\geq \mathbb{P} \sum\nolimits_i \{t_{i,n} \leq \tau + \epsilon\} \mathbb{P}_{i,n} \Delta_{i,n} \\
&\geq \mu_n H_\epsilon \\
&\geq \mathbb{P} \sum\nolimits_i \{t_{i,n} \leq \tau\} \mathbb{P}_{i,n} \Delta_{i,n} \geq \mathbb{P} S_\tau.
\end{aligned}$$

In the limit as $n \to \infty$ we get $\mathbb{P} S_{\tau + 2\epsilon} \geq \mu H_\epsilon \geq \mathbb{P} S_\tau$.

As $\epsilon \to 0$, the countable additivity of μ gives $\mu((0, \tau + \epsilon]] \to \mu((0, \tau]]$. Property[D] and right-continuity of the sample paths gives $\mathbb{P} S_{\tau + 2\epsilon} \to \mathbb{P} S_\tau$. It follows that $\mu((0, \tau]] = \mathbb{P} S_\tau$.

□

Problems

[1] Let $\{X_i : 0 \le i \le n\}$ be a submartingale with $X_0 \equiv 0$. For a fixed $\lambda \in \mathbb{R}^+$, define stopping times

$$\sigma := \min\{i : X_i \le -\lambda\} \wedge 1 \qquad \text{and} \qquad \tau := \min\{i : X_i \ge \lambda\} \wedge 1.$$

(i) Show that

$$\lambda \mathbb{P}\{\max_i X_i > \lambda\} \le \mathbb{P} X_\tau \{X_\tau \ge \lambda\} \le \mathbb{P} X_1 \{X_\tau \ge \lambda\} \le \mathbb{P}|X_n|.$$

(ii) Show that

$$\begin{aligned}\lambda \mathbb{P}\{\min_i X_i < -\lambda\} &\le \mathbb{P}(-X_\sigma)\{X_\sigma \le -\lambda\} \\ &\le -\mathbb{P} X_\sigma + \mathbb{P} X_n \{X_\sigma > -\lambda\} \le \mathbb{P}|X_n|.\end{aligned}$$

(iii) Suppose $\{Y_t : 0 \le t \le 1\}$ is a cadlag submartingale with $Y_0 \equiv 0$. Show that $\lambda \mathbb{P}\{\sup_t |Y_t| > \lambda\} \le 2\mathbb{P}|Y_1|$.

[2] Suppose a cadlag martingale $\{S_t : 0 \le t \le 1\}$, with $S_0 \equiv 0$, has a Doléans measure μ in the sense of Definition <2>, that is, $\mu((0, \tau]] = \mathbb{P} S_\tau$ for every $\tau \in \mathcal{T}_1$. Show that S has property [D] by following these steps.

(i) For a given $\tau \in \mathcal{T}_1$, let τ_n be the stopping time obtained by rounding up to the next integer multiple of 2^{-n}.

(ii) Invoke the Stopping Time Lemma to show that $0 \le \mathbb{P} S_{\tau_n}$ and $\mathbb{P} S_{\tau_n}^+ \le \mathbb{P} S_1^+$ for each $\tau \in \mathcal{T}_1$. Deduce that $\mathbb{P}|S_{\tau_n}| \le \kappa := 2\mathbb{P} S_1^+ < \infty$.

(iii) Invoke Fatou's lemma to show that $\sup_{\tau \in \mathcal{T}_1} \mathbb{P}|S_\tau| \le \kappa$.

(iv) For each $C \in \mathbb{R}^+$, show that

$$\mathbb{P} S_{\tau_n}\{S_{\tau_n} > C\} \le \mathbb{P} S_1 \{S_{\tau_n} > C\} \le \mathbb{P} S_1\{S_1 > \sqrt{C}\} + \kappa/\sqrt{C}.$$

Invoke Fatou, then deduce that $\sup_{\tau \in \mathcal{T}_1} \mathbb{P} S_\tau \{S_\tau > C\} \to 0$ as $C \to \infty$.

(v) Show that every cadlag function on $[0, 1]$ is bounded in absolute value. Deduce that the stopping time $\sigma_C := \inf\{t : S_t < -C\} \wedge 1$ has $\sigma_C(\omega) = 1$ for all C large enough (depending on ω). Deduce that $\mu((\sigma_C, 1]] \to 0$ as $C \to \infty$.

(vi) For a given $\tau \in \mathcal{T}_1$ and $C \in \mathbb{R}^+$, define $F_\tau := \{S_\tau < -C\}$. Show that $\tau' := \tau F_\tau^c + F_\tau$ is a stopping time for which

$$\mathbb{P}\left(S_1 - S_\tau\right) F_\tau = \mathbb{P}\left(S_{\tau'} - S_\tau\right) = \mu((\tau, \tau']] \le \mu((\sigma_C, 1]],$$

Hint: Show that if $\omega \in F_\tau$ then $\sigma_C(\omega) \le \tau(\omega)$ and if $\omega \in F_\tau^c$ then $\tau(\omega) = \tau'(\omega)$.

(vii) Deduce that $\sup_{\tau\in\mathcal{T}_1} \mathbb{P}\left(-S_\tau\right)\{S_\tau < -C\} \to 0$ as $C \to \infty$.

[3] Let $\{S_t : t \in \mathbb{R}^+\}$ be a submartingale of class [D]. Show that there exists an integrable random variable S_∞ for which $\mathbb{P}\left(S_\infty \mid \mathcal{F}_t\right) \ge S_t \to S_\infty$ almost surely and in L^1 by following these steps.

(i) Show that the uniformly integrable submartingale $\{S_n : n \in \mathbb{N}\}$ converges almost surely and in L^1 to an S_∞ for which $\mathbb{P}\left(S_\infty \mid \mathcal{F}_n\right) \ge S_n$.

(ii) For $t \le n$, show that $S_t \le \mathbb{P}\left(\mathbb{P}(S_\infty \mid \mathcal{F}_n) \mid \mathcal{F}_t\right) = \mathbb{P}\left(S_\infty \mid \mathcal{F}_t\right)$.

(iii) For $t \ge n$, show that

$$\mathbb{P}(S_t - S_n)^- \le \mathbb{P}(S_t - S_n)^+ \le \mathbb{P}(S_\infty - S_n)^+ \to 0 \qquad \text{as } n \to \infty.$$

(iv) For each $k \in \mathbb{N}$, choose $n(k)$ for which $\mathbb{P}|S_\infty - S_{n(k)}| \le 4^{-k}$. Invoke Problem [1] to show that $\sum_k \mathbb{P}\{\sup_{t\ge n(k)} |S_t - S_{n(k)}| > 2^{-k}\} < \infty$.

(v) Deduce that $S_t \to S_\infty$ almost surely.

C.3 Notes

My exposition in this Chapter is based on ideas drawn from a study of Métivier (1982, §13), Dellacherie and Meyer (1982, Chapter VII), and Chung and Williams (1990, Chapter 2). The construction in Section 2 appears new, although it is clearly closely related to existing methods.

References

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