Project 1

Do stochastic processes exist?

If you wish to take this course for credit, you should keep a notebook that contains detailed proofs of the results sketched in my handouts. You may consult any texts you wish and you may ask me or anyone else as many questions as you like.

Please do not just copy out standard proofs without understanding. Please do not just copy from someone else's notebook.

I suggest you work in groups to figure out the proofs. You should arrange to meet with me in small groups every week to discuss any difficulties you have with producing an account in your own words. I will also point out refinements, if you are interested.

At the end of the semester, I will look at your notebook to make up a grade. By that time, you should have a pretty good written account of a significant chunk of stochastic calculus.

I will be writing these projects in note form, without worrying too much about grammar or about making proper sentences.

1.1 Different ways to think about a stochastic process

Index set T. For example, $T = \mathbb{N}$ or $\{0, 1, \dots, N\}$ (discrete time) or $T = [0, 1]$ or $\mathbb{R}$ or $\mathbb{R}^+$ (continuous time).

(i) A set of real-valued random variables $\{X_t : t \in T\}$ all living on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Often require X_t to be $\mathcal{F}_t$-measurable, for some filtration $\{\mathcal{F}_t : t \in T\}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s < t$. That is, X is ***adapted to the filtration***.

(ii) A map $X : T \times \Omega \to \mathbb{R}$. Often require measurability with respect to some sigma-field on the product space. For example, if $T = \mathbb{R}^+$ then X is said to be ***progressively measurable*** if the restriction of X to $[0, t] \times \Omega$ is $\mathcal{B}[0, t] \otimes \mathcal{F}_t$-measurable for each fixed t in $\mathbb{R}^+$.

(iii) The sample path $X(\cdot, \omega)$ is an element of $\mathbb{R}^T$, for each fixed ω. Thus $X : \Omega \to \mathbb{R}^T$. The cylinder sigma-field (a.k.a product sigma-field) $\mathcal{F}_T$ on $\mathbb{R}^T$ is the smallest sigma-field for each of the coordinate projections, $\pi_t : x \mapsto x(t)$ is $\mathcal{F}_T\backslash\mathcal{B}(\mathbb{R})$-measurable. Measurability of X_t, for each fixed t, implies X is $\mathcal{F}\backslash\mathcal{F}_T$-measurable. Why? Often want sample paths to concentrate in some subset of $\mathbb{R}^T$, such as $C(T)$, the set of continuous real functions on T.

1.2 Existence

We need a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for which each X_t is an $\mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurable random variable. The Daniell-Kolmogorov theorem solves the problem by taking $\Omega = \mathbb{R}^T$ with $X_t(\omega) = \omega(t)$ for each t and each $\omega \in \mathbb{R}^T$.

Start from prescribed finite-dimensional distributions (***fidis***). That is, for each finite subset J of T we want the random vector $X_J := (X_t : t \in J)$ to have distribution P_J, a prescribed probability measure on $\mathcal{B}(\mathbb{R}^J)$. That is, we want $\mathbb{P}$ to have the property

$$\mathbb{P}\{\omega : X_J(\omega) \in B\} = P_J(B) \qquad \text{for each } B \in \mathcal{B}(\mathbb{R}^J).$$

If such a $\mathbb{P}$ is to exist we need consistent fidis: if $J' \supset J$ then

<1> $$P_J B = P_{J'}(B \times \mathbb{R}^{J'\backslash J}) \qquad \text{for each } B \in \mathcal{B}(\mathbb{R}^J).$$

Put another way, we need $P_J = \pi_{J'J} P_{J'}$ (image measure), where $\pi_{J'J}$ is the projection from $\mathbb{R}^{J'}$ onto $\mathbb{R}^J$.

The subset $\mathcal{C}_J$ of $\mathcal{F}_T$ consists of all cylinder sets with a base in J, that is, sets of the form $A \times \mathbb{R}^{T\backslash J}$ with $A \in \mathcal{B}(\mathbb{R}^J)$. Define $\mathcal{C} = \cup_{J \text{ finite}} \mathcal{C}_J$. Explain why $\mathcal{C}$ is a field of subsets on Ω with $\mathcal{F}_T = \sigma(\mathcal{C})$. Define $\mathbb{P}$ on $\mathcal{C}$ by

<2> $$\mathbb{P}C = P_J A \qquad \text{if } C = A \times \mathbb{R}^{T\backslash J}$$

Why do we need <1> to ensure that $\mathbb{P}$ is well defined?

Explain why $\mathbb{P}$ is a finitely additive measure with $\mathbb{P}\Omega = 1$. Now invoke a standard measure theory result. Compare with Folland (1999, Section 1.4).

<3> **Theorem.** *A finitely additive measure $\mathbb{P}$ on the field $\mathcal{C}$ has an extension to a countably additive measure on $\sigma(\mathcal{C})$ if and only if it has the following property: $\mathbb{P}C_n \downarrow 0$ for each decreasing sequence $\{C_n\}$ in $\mathcal{C}$ with $\cap_n C_n = \emptyset$.*

1.2.1 Case: $T = \mathbb{N}$

Explain the following.

(i) Consider a decreasing sequence of cylinder sets C_n. It is enough to show that $\inf_n \mathbb{P}C_n > 0$ implies $\cap_n C_n \neq \emptyset$.

(ii) Without loss of generality we may assume $C_n = A_n \times \mathbb{R}^{\mathbb{N}\backslash J_n}$ where $J_n = \{1, 2, \ldots, n\}$ and $A_n \in \mathcal{B}(\mathbb{R}^{J_n})$.

(iii) Without loss of generality we may assume each A_n is compact. See Pollard (2001, Problem 2.12) for approximation of Borel sets from inside by compact sets.

(iv) Each A_n must be nonempty. Thus there exist points $(x_{j,1}, x_{j,2}, \dots, x_{j,j})$ in A_j for each $j \in \mathbb{N}$. For each k, the points

$$(x_{j,1}, \dots, x_{j,k}) \text{ belong to } A_k \text{ for all } j \geq k$$

(v) Inductively choose infinite subsets $\mathbb{N}_1 \supseteq \mathbb{N}_2 \supseteq \mathbb{N}_3 \dots$ of $\mathbb{N}$ such that there exist $z_1, z_2, \dots$ for which

$$(x_{j,1}, \dots, x_{j,k}) \to (z_1, \dots, z_k) \in A_k \qquad \text{along } \mathbb{N}_k\text{, for each } k$$

(vi) Conclude that $z := (z_1, z_2, \dots) \in \cap_n C_n$.

1.2.2 Case : general T

(i) Explain why the argument just given also applies to any countable T.

(ii) Let $\mathcal{F}_S$ be the smallest sigma-field on Ω for which each coordinate projection X_t, for t in S, is $\mathcal{F}_S\backslash\mathcal{B}(\mathbb{R})$-measurable. Explain why the $\mathbb{P}$ defined in <2> is a countably additive probability measure when restricted to $\mathcal{F}_S$, for each countable subset S of T.

Explain why $\mathcal{F}_T = \cup_S \mathcal{F}_S$, the union running over all countable subsets S of T.

(iii) For each set B in $\mathcal{F}_T$ there exists a countable S (depending on B) such that $B \in \mathcal{F}_S$. Show that

$$\mathcal{A}_S = \{B \in \mathcal{F}_S : \text{ if } \omega \in B \text{ and } \omega'\big|_S = \omega\big|_S \text{ then } \omega' \in B\}$$

is a sigma-field that contains every cylinder set with base $J \subset S$. Deduce that $\mathcal{A}_S = \mathcal{F}_S$.

(iv) For $T = [0, 1]$, explain why $C(T)$, the set of all bounded continuous real functions on T, cannot belong to the sigma-field $\mathcal{F}_T$.

1.3 Brownian motion with continuous sample paths

A Brownian motion indexed by a subset T of the real line is a zero mean Gaussian process $\{B_t : t \in T\}$ with $\mathrm{cov}(B_s, B_t) = \min(s,t)$. (Equivalently, it has independent increments and $B_t - B_s \sim N(0, t-s)$ for $s < t$.)

Show that there exists a Brownian motion indexed by $T = [0,1]$ with continuous sample paths by the following steps.

(i) Define $T_k = \{j/2^k : j = 0, 1, \ldots, 2^k\}$ and $S = \cup_{k\in\mathbb{N}} T_k$ (the dyadic rationals). Use the Daniell-Kolmogorov theorem to construct a probability measure $\mathbb{P}$ on the cylinder sigma-field of $\mathbb{R}^s$ such that the coordinate maps $\{X_s : s \in S\}$ define a Brownian motion indexed by S.

(ii) Fix integers $k < \ell$. Use a reflection argument (cf. Pollard 2001, Section 4.6) to show that

$$\mathbb{P}\{\max_{s\in T_\ell,\, 0\le s\le 2^{-k}} |X_s| > x\} \le 2\mathbb{P}\{|X_{2^{-k}}| > x\} \le 2\exp(-\tfrac{1}{2}x^2/2^{-k}).$$

(iii) Define $t(k,j) = j/2^k$. Define

$$M_{k,j}(\omega) := \sup\{|X_s(\omega) - X_{t(k,j)}(\omega)| : t(k,j) \le s \le t(k,j+1)\}$$

for $j = 0, 1, \ldots, 2^k - 1$. Show that

$$\mathbb{P}\{M_{k,0} > x\} \le 2\exp(-\tfrac{1}{2}x^2/2^{-k})$$

(iv) Define $H_k := \{\max_j M_{k,j} > x_k\}$ with $x_k := c\sqrt{k2^{-k}}$ for some positive constant c. Show that

$$\mathbb{P}H_k < e^{-k} \qquad \text{if } c \text{ is large enough.}$$

(v) Use Borel-Cantelli to deduce that the set $\Omega_0 := \{H_k \text{ i.o.}\}^c$ has $\mathbb{P}$ measure 1.

(vi) Explain why, for each $\omega \in \Omega_0$, there exists a finite $k_0(\omega)$ such that $\max_j M_{k,j}(\omega) \le x_k$ for all $k \ge k_0(\omega)$.

(vii) Define $B_1(\omega) := X_1(\omega)$ and for each $t \in [0,1)$,

$$B_t(\omega) := \lim_{m\to\infty} \sup\nolimits_{t<s\le t+1/m,\, s\in S} X_s(\omega)$$

Explain why $B_t(\omega) = \lim_{s\downarrow\downarrow t} X_s(\omega)$ if $\omega \in \Omega_0$ and $t < 1$. *(The notation $s \downarrow\downarrow t$ means: s decreases to t through the set $S \cap (t,1]$.)*

(viii) Explain $t \mapsto B_t(\omega)$ is continuous for each $\omega \in \Omega_0$.

(ix) Explain why $\{B_t(\omega) : 0 \le t \le 1,\ \omega \in \Omega_0\}$ is a Brownian motion with continuous sample paths.

(x) [Harder] Explain why, for each $\omega \in \Omega_0$, there exists a finite constant $C_0(\omega)$ such that

$$\sup\{|B_t(\omega) - B_{t'}(\omega)| : |t - t'| < \delta\} \le C_0(\omega)\sqrt{\delta \log(1/\delta)} \qquad \text{for all } 0 < \delta \le 1.$$

References

Folland, G. B. (1999). *Real Analysis: Modern Techniques and Their Applications*. Wiley.

Pollard, D. (2001). *A User's Guide to Measure Theoretic Probability*. Cambridge University Press.