Project 10

Prediction and predictability

This Project introduces a circle of ideas related to the problem of prediction based on past information, including the definitions of predictable processes and predictable stopping times.

10.1 Notation

Let $\mathcal{T}$ denote the set of all stopping times for a given standard filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$ on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As before, define $\mathfrak{S} := \Omega \times \mathbb{R}^+$ and $\mathfrak{S}^\circ := \Omega \times (0, \infty)$ and assume that the filtration is standard. Write π for the projection map, $(\omega, t) \mapsto \omega$, from $\mathfrak{S}$ onto Ω.

I am beginning to find the problems with predictable processes at 0 a nuisance. Up to now I have been guided by a remark of Dellacherie and Meyer (1978, page 121), to the effect that the predictable sigma-field should live on $\mathfrak{S}^\circ$, even though they did not adopt this suggestion themselves. Rogers and Williams (1987, Section IV.6) did make predictable ("previsible" in their terminology) processes and the predictable sigma-field live on $\mathfrak{S}^\circ$, at the cost of some subtleties in later definitions. For example, in their Section VI.12 they restricted predictable stopping times to be everywhere strictly positive, a cunning requirement that eliminates many of the difficulties that I encountered when first attempting to write this Project.

Accordingly, I will now work with a sigma-field $\overline{\mathcal{P}}$ on $\mathfrak{S}$ and say that a process on $\mathfrak{S}$ is predictable if it is $\overline{\mathcal{P}}$-measurable. The new sigma-field will be generated by all sets in $\mathcal{P}$ together with all sets of the form $F \times \{0\}$, with $F \in \mathcal{F}_0$, but that is not really the most useful characterization.

Lect 20, Wednesday 1 April

10.2 The (new) predictable sigma-field

You should modify the arguments from Section 6.4 to show that each of the following collections of sets or processes generate the same sigma-field on $\mathfrak{S}$. I will write $\overline{\mathcal{P}}$ for that sigma-field.

(i) the collection of all sets of the form $F \times (a, b]$ with $0 \le a < b < \infty$ and $F \in \mathcal{F}_a$ or of the form $F \times \{0\}$ with $F \in \mathcal{F}_0$

(ii) the set of all stochastic intervals $[[0,\tau]] := \{(\omega,t) \in \mathfrak{S} : 0 \le t \le \tau(\omega)\}$ for $\tau \in \mathcal{T}$

(iii) the set of all adapted processes on $\mathfrak{S}$ whose sample paths are continuous from the left at each t in $(0,\infty)$

(iv) the set $\mathbb{C}$ of all adapted processes on $\mathfrak{S}$ with continuous sample paths

(v) the set $\mathcal{Z}$ of all "zero sets" of the form $Z = \{(\omega,t) \in \mathfrak{S} : X(\omega,t) = 0\}$ for some X in $\mathbb{C}$

For (v) argue as follows.

(a) The zero sets all belong to the sigma-field generated by $\mathbb{C}$.

(b) Each nonnegative $\mathbb{C}$-process X is expressible as a pointwise limit of simple functions, $2^{-n}\sum_{i=1}^{4^n}\{X \ge i2^{-n}\}$, and

$$\{X \ge i2^{-n}\} = \{(i2^{-n} - X)^+ = 0\} \in \mathcal{Z}.$$

Remark. The generating class $\mathbb{C}$ is contained in the set of all R-processes, which implies that $\overline{\mathcal{P}}$ is a sub-sigma-field of $\mathcal{O}$, the optional sigma-field on $\mathfrak{S}$. As shown in Section 4.4, every optional process is progressively measurable.

You should also establish these facts.

(a) The generating class $\mathcal{Z}$ is stable under the formation of finite unions and countable intersections—a $(\cup f, \cap c)$-paving on $\mathfrak{S}$. Hint: Why may we assume $0 \le X \le 1$? For a sequence $\{X_i\}$ of such processes, consider the sets $\{\min_{i \le n} X_i = 0\}$ and $\{\sum_{i\in\mathbb{N}} 2^{-i}X_i = 0\}$.

(b) If $Z \in \mathcal{Z}$ then Z^c can be written as a countable union of $\mathcal{Z}$-sets. Hint: Consider $\{|X| \ge n^{-1}\}$.

(c) Every finite measure μ on $\mathcal{P}$ is $\mathcal{Z}$-inner regular, that is,

$$\mu A = \sup\{\mu Z : A \supseteq Z \in \mathcal{Z}\} \qquad \text{for each } A \in \mathcal{P}.$$

Hint: Consider the collection $\mathcal{P}_0$ of all sets A in $\overline{\mathcal{P}}$ for which both A and A^c have the desired property.

10.3 Foretelling of predictable stopping times

In Project 5 you got a taste of the benefits of predictability (albeit in discrete time) when we stopped the sum of conditional variances just before it got too big. In continuous time the corresponding idea is captured by the concept of a foretellable stopping time.

<1> **Definition.** *A sequence of stopping times $\{\tau_n\}$ is said to* ***foretell*** *(or announce) τ if $\tau_n \le \tau$ and $\tau_n \uparrow\uparrow \tau$ everywhere on $\{\tau > 0\}$.*

> **Remark.** Note that τ is necessarily a stopping time, because $\{\tau \le t\} = \cup\{\tau_n \le t\} \in \mathcal{F}_t$ for all t in $\mathbb{R}^+$.
>
> Of course $\tau_n = 0$ on $\{\tau = 0\}$, so it would be unreasonable to demand that $\tau_n < \tau$ everywhere, rather than just on the set $\{\tau > 0\}$.
>
> For a foretelling sequence, it is equivalent to show that there exists a countable set of stopping times $\{\sigma_i : i \in I\}$ for which $\sup_i \sigma_i(\omega) = \tau(\omega)$ for each ω and $\sigma_i(\omega) < \tau(\omega)$ for every i if $\tau(\omega) > 0$: the sequence $\tau_k = \max\{\sigma_i : i \in I_k\}$ has the desired property if the I_k are finite and $I_k \uparrow I$ as $k \to \infty$.

<2> **Example.** Let τ be the debut of a zero set $Z = \{X = 0\}$, where $X \in \mathbb{C}$. That is, $\tau(\omega) = \inf\{t \in \mathbb{R}^+ : X(\omega, t) = 0\}$. In fact that infimum is achieved because $\{t \in \mathbb{R}^+ : X(\omega, t) = 0\}$ is closed.

Similarly, each debut $\tau_n(\omega) := n \wedge \inf\{t : |X(t, \omega)| \le 1/n\}$ is also a stopping time with $\tau_n(\omega) \le \tau(\omega)$ for every ω.

The times $\{\tau_n\}$ foretell τ. Indeed, if $0 < \epsilon \le \tau(\omega) < \infty$ then $|X(\omega, \cdot)|$ is bounded away from zero for $0 \le t \le \tau(\omega) - \epsilon$, implying $\tau_n(\omega) \ge \tau(\omega) - \epsilon$ eventually. Moreover, we cannot have $\tau_n(\omega) = \tau(\omega) > 0$, for otherwise $X(\omega, \cdot)$ woud have a jump at $\tau(\omega)$.

Similarly, if $\tau(\omega) = \infty$ and if C is a finite constant then $|X(\omega, \cdot)|$ is bounded away from zero for $0 \le t \le C$, implying $n \ge \tau_n(\omega) \ge C$ eventually.

□

In an obvious sense, a stopping time τ for which there exists a foretelling sequence can be predicted from past information. Many authors would define such a τ to be a ***predictable stopping time***, but a slightly different definition has some advantages.

<3> **Definition.** *A random variable $\tau : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ is said to be a* ***predictable stopping time*** *if $[[\tau, \infty)) \in \mathcal{P}$.*

Note that the defining property of a predictable stopping time is stronger than the assertion that $[[\tau,\infty))$ is adapted, which is equivalent to the assertion that τ is a stopping time.

Remark. If τ is a predictable stopping time then $[[\tau]] = [[0,\tau]] \cap [[\tau,\infty))$, a predictable set. Conversely, if $\tau : \Omega \to [0,\infty]$ and $[[\tau]] \in \overline{\mathcal{P}}$ then

$$\{\omega : \tau(\omega) \le t\} = \pi\left([[\tau]] \cap \Omega \times [0,t]\right) \in \mathcal{F}_t \qquad \text{for each } t \in \mathbb{R}^+,$$

so that τ is necessarily a stopping time. Moreover, $[[\tau,\infty)) = [[\tau]] \cup ((\tau,\infty))$, which belongs to $\overline{\mathcal{P}}$ because $((\tau,\infty))$ is an adapted process—the filtration is standard—with left-continuous sample paths.

If τ has a foretelling sequence of stopping times then it must be predictable, in the sense of Definition <3>, because

$$[[\tau,\infty)) = \left(\{\omega : \tau(\omega) = 0\} \times \mathbb{R}^+\right) \cup \bigcap\nolimits_{n\in\mathbb{N}} ((\tau_n,\infty)) \in \overline{\mathcal{P}}$$

For a standard filtration (as assumed throughout this Project), predictability in the sense of Definition <3> also implies existence of a foretelling sequence, as shown by the next Theorem (borrowed from Métivier 1982, page 25). See Dellacherie and Meyer (1978, IV.77) for the slightly more delicate assertion for nonstandard filtrations.

<4> **Theorem.** *For a standard filtration, a stopping time has a foretelling sequence if it is predictable.*

PROOF Initially suppose τ is bounded by a constant C.

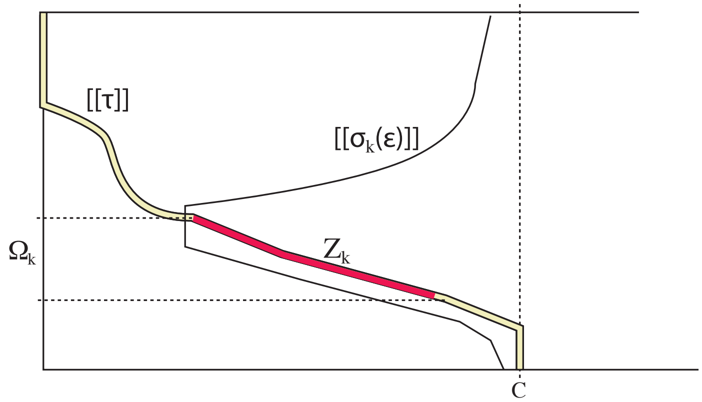
(i) Show that the map $\omega \mapsto (\omega, \tau(\omega))$ is $\mathcal{F}\backslash\overline{\mathcal{P}}$-measurable.

(ii) Define the probability measure μ_τ on $\overline{\mathcal{P}}$ to be the image of $\mathbb{P}$ under the map from (i). Show that μ_τ concentrates on $[[\tau]]$. Hint: $\mu_\tau f = \mathbb{P}^\omega f(\omega,\tau(\omega))$ at least for predictable, nonnegative f.

(iii) Show that there exists an increasing sequence of zero sets $Z_k \subseteq [[\tau]]$ with $\mu_\tau Z_k \uparrow 1$. Let $\Omega_k = \pi Z_k$. Deduce that $\mathbb{P}\Omega_k \uparrow 1$. Write N_1 for the $\mathbb{P}$-negligible set $\cap_k \Omega_k^c$.

(iv) Define $\tau_k := C \wedge (\text{the debut of } Z_k)$. Show that

$$\tau_k(\omega) = \tau(\omega)\{\omega \in \Omega_k\} + C\{\omega \notin \Omega_k\}.$$



(v) Invoke Example <2> to find a foretelling sequence $\{\tau_{k,i} : i \in \mathbb{N}\}$ for τ_k. For each fixed $\epsilon > 0$ choose $\sigma_k(\epsilon) = \tau_{k,i}$ for an i large enough that $\mathbb{P}\{\sigma_k(\epsilon) \le \tau_k - \epsilon\} < \epsilon/2^k$. Explain why $\sigma_k(\epsilon) < \tau$ on $\Omega_k\{\tau > 0\}$.

(vi) Define $\sigma(\epsilon) := \tau \wedge \inf_k \sigma_k(\epsilon)$. Show that $\sigma(\epsilon) < \tau$ on $N_1^c\{\tau > 0\}$ and $\mathbb{P}\{\sigma(\epsilon) < \tau - \epsilon\} < \epsilon$.

(vii) Define $\sigma := \sup_{j\in\mathbb{N}} \sigma(2^{-j})$. Show that $\sigma = \tau$ on the complement of some $\mathbb{P}$-negligible set N_2.

(viii) Redefine $\sigma(2^{-j})$ to equal $j \wedge \left(\tau - j^{-1}\right)^+$ on $N_1 \cup N_2$. Show that the $\sigma(2^{-j})$'s then define a fortelling sequence of stopping times for τ.

(ix) Relax the assumption that τ is bounded by a constant. Argue as above for each $\tau \wedge n$, then combine the resulting countable collection of foretelling times into a single foretelling sequence.

□

Both characterizations of predictable stopping times are useful.

<5> **Example.** (due to Lepingle? see Dellacherie and Meyer 1982, VIII.11) Let τ be the debut of a predictable set A. For concreteness suppose $A = \{(\omega, t) \in \mathfrak{S} : |H(\omega, t)| > C\}$ where H is a predictable process and C is a finite constant.

If $(\omega, \tau(\omega) \in A$ whenever $\tau(\omega) < \infty$ (that is, if $[[\tau]] \subset A$) then τ is a predictable stopping time, because $[[\tau]] = [[0, \tau]] \cap A \in \overline{\mathcal{P}}$. We could foretell τ by a sequence of stopping times τ_k with $(\omega, \tau_k(\omega) \notin A$ whenever $\tau(\omega) > 0$, thereby avoiding any nasty jumps in H at time τ.

If $[[\tau]]$ is not completely contained in A the stopping time τ might not be predictable. Nevertheless, it is still possible to replace τ by a sequence of stopping times $\tau_k \le \tau$ for which $\sup_k \tau_k = \tau$ and for which $\tau_k(\omega) < \tau(\omega)$ on the set $\{\omega : \tau(\omega) > 0,\ (\omega, \tau(\omega)) \in A\}$. We would then have $|H|((0, \tau_k]] \le C$, again avoiding any nasty jumps in H at time τ.

To construct such τ_k, let σ be the debut of the predictable set $D = A \cap [[0, \tau]]$. Then $[[\sigma]] = D \in \overline{\mathcal{P}}$, which shows that σ is a predictable stopping time. It has a foretelling sequence of stopping times σ_k. The sequence $\tau_k = \tau \wedge \sigma_k$ has the desired property.

□

<6> **Example.** (compare with Dellacherie and Meyer 1982, page 323) In Section 2 of Project 8, I asserted that if $X \in \mathbb{SMG}$ and if $\{H^{(n)} : n \in \mathbb{N}\}$ is a sequence of locally bounded predictable processes that converges ucpc to zero then $H^{(n)} \bullet X \xrightarrow{uccp} 0$. The assertion will follow from by subsequencing argument, which will show that there exists a subsequence $\{n_k : k \in \mathbb{N}\}$ and a $\mathbb{P}$-negligible set N for which $\{\omega \in N^c\}H^{(n_k)}(\omega, s)$ is both locally uniformly bounded and poitwise convergent to zero. Note that we will then have $H^{(n_k)} \bullet X \xrightarrow{uccp} 0$.

(a) Show that for each $(\omega, t) \in \mathfrak{S}$ there exists a finite constant $C_n(\omega, t)$ such that $\sup_{0 \le s \le t} |H^{(n)}(\omega, s)| \le C_n(\omega, t)$. Hint: $H^{(n)}$ is locally bounded.

(b) Explain why, for each k in $\mathbb{N}$, there exists an n_k such that

$$\mathbb{P}\{\sup_{0 \le s \le k} |H^{(n)}(\omega, s)| > 2^{-k}\} < 2^{-k} \qquad \text{for all } n \ge n_k.$$

Deduce that there exists a $\mathbb{P}$-negligible set N and finite constants $C(\omega, t)$ such that

$$\sum\nolimits_{k \in \mathbb{N}} \sup_{0 \le s \le t} |H^{(n_k)}(\omega, s)| \le C(\omega, t) < \infty \qquad \text{for each } \omega \in N^c.$$

(c) Define $Z(\omega, s) := \sum_{k \in \mathbb{N}} |H^{(n_k)}(\omega, s)\{\omega \in N^c\}|$. For each m in $\mathbb{N}$ let τ_m denote the debut of the (predictable) set $\{Z > m\}$. Explain why $\tau_m(\omega) \to \infty$ as $m \to \infty$ for each ω.

(d) Use Example <5> to show that there exist stopping times $\{\tau_{m,i} : i \in \mathbb{N}\}$ for which $\sup_i \tau_{m,i} = \tau_m$ and $\sup_{s \in \mathbb{R}^+} Z(\omega, s)\{0 \le s \le \tau_{m,i}(\omega)\} \le m$ for each i and m.

(e) Complete the argument.

Now consider the problem posed at the start of the Example. If $H^{(n)} \bullet X$ did not converge uccp to zero there would exist some $t \in \mathbb{R}^+$ and some subsequence along which $\mathbb{P}\{\sup_{s \le t} |H^{(n)} \bullet X_s| > \epsilon\} > \epsilon$ for some $\epsilon > 0$. Argue for a contradiction along a sub-subsequence.

□

Lect 21, Monday 5 April

10.4 The pre-τ sigma-field

In discrete time it is fairly obvious how to define a sigma-field $\mathcal{F}(\tau-)$ that makes rigorous the concept of *information available prior to time* τ. If $\tau \equiv n_0 > 0$, a constant, then $\mathcal{F}(\tau-) = \mathcal{F}_{n_0-1}$. If τ is a stopping time, then on the set $\{\tau = n\}$ the information should correspond to $\mathcal{F}_{n-1}$, which is almost equivalent to taking $\mathcal{F}(\tau-)$ to be the sigma-field generated by sets of the form $F\{m < \tau\}$ with $F \in \mathcal{F}_m$. There is a slight problem about how to define the past prior to time 0, which we can avoid by imagining the filtration extended back into the past by defining $\mathcal{F}_{-n} = \mathcal{F}_0$ for $n \in \mathbb{N}$. With that convention, $\mathcal{F}_0$ is its own past and all $\mathcal{F}_0$-measurable sets belong to $\mathcal{F}(\tau-)$. A similar definition makes sense in continuous time.

<7> **Definition.** *For a stopping time τ define the pre-τ sigma-field $\mathcal{F}(\tau-)$ by means of its class of generating sets: all sets in $\mathcal{F}_0$ together with all sets of the form $F\{s < \tau\}$ with $s \in \mathbb{R}^+$ and $F \in \mathcal{F}_s$.*

What properties should we expect of $\mathcal{F}(\tau-)$? Which random variables should be $\mathcal{F}(\tau-)$-measurable?

<8> **Example.** If $\tau \equiv t_0$, a constant, then $\mathcal{F}(\tau-) = \sigma\{\mathcal{F}_s : s < t_0\}$, as one might hope.

□

<9> **Example.** The stopping time τ itself is $\mathcal{F}(\tau-)$-measurable, because every set $\{s < \tau\}$, for $s \in \mathbb{R}^+$, is one of the generators for the sigma-field. In particular, $\{\tau < \infty\} \in \mathcal{F}(\tau-)$.

My intuition has a hard time in accepting, in general, that τ should somehow be determined by information before time τ.

□

<10> **Example.** Suppose τ is a predictable stopping time with a foretelling sequence $\{\tau_n\}$. Intuitively, any information obtained strictly before time τ should be available at one of the τ_n times. Formalize this intuition, by showing that $\mathcal{F}(\tau-) = \sigma\left(\cup_{n \in \mathbb{N}} \mathcal{F}(\tau_n)\right)$.

(i) Show that $\cup_{n\in\mathbb{N}}\mathcal{F}(\tau_n)$ contains all the generators for $\mathcal{F}(\tau-)$. Hint: Show that $F\{s<\tau_n\} \uparrow F\{s<\tau\}$ and $F\{s<\tau_n\}\{\tau_n \le t\} \in \mathcal{F}_t$ if $F \in \mathcal{F}_s$.

(ii) Show that $\mathcal{F}(\tau_n) \subseteq \mathcal{F}(\tau-)$ for every n. Hint: Break an F in $\mathcal{F}(\tau_n)$ into two parts, $F = F\{\tau = 0\} \cup F\{0 < \tau\}$. The first contribution belongs to $\mathcal{F}_0$. Express the second contribution as a countable union of sets of the form $F\{\tau_n \le t < \tau\}$.

□

<11> **Example.** For which processes X should $Z(\omega) := X(\omega, \tau(\omega))\{\tau(\omega) < \infty\}$ be $\mathcal{F}(\tau-)$-measurable? If τ itself is predictable then Z would seem to depend only on information prior to time τ. One might therefore expect that the measurability property should hold at least for X in the class $\mathbb{L}$ of adapted processes with left-continuous sample paths. In fact it also holds for all predictable X.

(i) First suppose that $\{X_t : t \in \mathbb{R}^+\} \in \mathbb{L}$. Show that Z is a pointwise limit of random variables

$$Z_n(\omega) := X(\omega, 0)\{\tau(\omega) = 0\} + \sum_{k\in\mathbb{N}} X\left(\omega, \frac{k-1}{2^n}\right)\left\{\frac{k-1}{2^n} < \tau(\omega) \le \frac{k}{2^n}\right\}.$$

(ii) Explain why each summand in the last sum is $\mathcal{F}(\tau-)$-measurable. Hint: Approximate $X(\omega, s)\{s < \tau(\omega)\}$ by linear combinations of $\mathcal{F}(\tau-)$ generators.

(iii) Deduce that Z is $\mathcal{F}(\tau-)$-measurable.

(iv) Use a lambda-space argument to extend the measurability property to all predictable processes. Hint: First consider the set $\mathcal{H}$ of all bounded predictable processes with the desired property. Note that $\mathcal{H} \supset \mathbb{L}$.

□

10.5 Predictable cross-sections

The following deep measure theoretic result was proved in Appendix B. The first part of the theorem was used in Project 4 to establish that the debut of a progressively measurable set is a stopping time (if the filtration is standard).

<12> **Theorem.** *Suppose* $(\Omega, \mathcal{G}, \mathbb{P})$ *is a complete probability space. Let* π *denote the map that projects* $\mathfrak{S} := \Omega \times \mathbb{R}^+$ *onto* Ω*. If* A *is a* $\mathcal{G} \otimes \mathcal{B}(\mathbb{R}^+)$*-measurable subset of* $\overline{\mathfrak{S}}$ *then*

(i) $\pi A \in \mathcal{G}$

(ii) *there exists a* ***measurable cross-section*** *of* A*, that is, a* $\mathcal{G}$*-measurable random variable* $\psi : \Omega \to [0, \infty]$ *such that* $\psi(\omega) = +\infty$ *if* $\omega \notin \pi A$ *and* $(\omega, \psi(\omega)) \in A$ *for* $\mathbb{P}$*-almost all* ω *in* πA*.*

Remember that the graph of a function $\psi : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ is defined as

$$[\![\psi]\!] := \{(\omega, t) \in \mathfrak{S} : t = \psi(\omega) < \infty\}.$$

The cross-section property of the ψ in (ii) implies that $\pi\left([\![\psi]\!] \backslash A\right)$ is a $\mathbb{P}$-negligible set. It also implies that the measure $\mathbb{Q}_\psi$ defined on the product sigma-field by

<13> $$\mathbb{Q}_\psi f(\omega, t) = \mathbb{P}\{\psi(\omega) < \infty\} f(\omega, \psi(\omega)) \qquad \text{for } f \in \mathcal{M}^+(\mathfrak{S})$$

concentrates on A. That is, $\mathbb{Q}_\psi A^c = 0$.

If A happens to be a predictable set we can do slightly better, by making the sectioning variable a predictable stopping time, but only if we are prepared to relax the property that almost every point of πA gets mapped to a point of A. The relaxation will have little effect on the usefulness of the stopping time.

<14> **Theorem.** *Let* A *be a predictable subset of* $\mathbb{R}^+ \otimes \Omega$ *for a standard filtration. For each* $\epsilon > 0$ *there exists a predictable stopping time* τ_ϵ *such that* $[\![\tau_\epsilon]\!] \subseteq A$ *and* $\mathbb{P}\{\tau_\epsilon < \infty\} > \mathbb{P}\pi A - \epsilon$*.*

Proof Let ψ be a measurable cross-section for A. Write $\mathbb{Q}$ for the restriction to $\overline{\mathcal{P}}$ of the $\mathbb{Q}_\psi$ defined in <13>. Given an $\epsilon > 0$, invoke the $\mathcal{Z}$-inner regularity of $\mathbb{Q}$ to find a zero set $Z_\epsilon = \{X_\epsilon = 0\} \subseteq A$, with $X_\epsilon \in \mathbb{C}$, such that $\mathbb{Q}Z_\epsilon > \mathbb{Q}A - \epsilon$.

Write τ_ϵ for the debut of Z_ϵ. By Example <2>, it is a predictable stopping time for which $[\![\tau_\epsilon]\!] \subseteq Z_\epsilon \subseteq A$. The set $\{\tau_\epsilon < \infty\}$ equals πZ_ϵ, the projection of Z_ϵ onto Ω, and

$$\begin{aligned}
\mathbb{P}\pi Z_\epsilon &= \mathbb{P}\{\omega : (\omega, t) \in Z_\epsilon \text{ for some } t \in \mathbb{R}^+\} \\
&\geq \mathbb{P}\{\omega : (\omega, \psi(\omega)) \in Z_\epsilon\} \\
&= \mathbb{Q}Z_\epsilon \\
&\geq \mathbb{P}\pi A - \epsilon,
\end{aligned}$$

as desired.

□

<15> **Example.** If X and Y are nonnegative predictable processes for which

<16> $$\mathbb{P}X(\tau)\{\tau < \infty\} \le \mathbb{P}Y(\tau)\{\tau < \infty\}$$

for every predictable stopping time, show that $X \le Y$ as-p'wise.

We need to show that $\mathbb{P}\pi\{X > Y\} = 0$. Break the set $\{X > Y\}$ into a contable union of predicatble sets $A_\lambda = \{X > \lambda \ge Y\}$, with λ ranging over positive rational numbers. If $\mathbb{P}\pi A_\lambda > 0$, there exists a predictable stopping time τ_λ such that $[[\tau_\lambda]] \subseteq A_\lambda$ and $\mathbb{P}\{\tau_\lambda < \infty\} \ge \mathbb{P}\pi A_\lambda/2 > 0$. Then we would have

$$\mathbb{P}X(\tau_\lambda)\{\tau_\lambda < \infty\} > \lambda\mathbb{P}\{\tau_\lambda < \infty\} \ge \lambda\mathbb{P}Y(\tau_\lambda)\{\tau_\lambda < \infty\},$$

which contradicts <16>.

If the inequality in <16> is replaced by an equality, then a similar argument shows that $Y \le X$ as-p'wise, and consequently $X = Y$ as-p'wise.

□

10.6 The jumps of a cadlag process

By definition, a cadlag function $x(\cdot)$ on $\mathbb{R}^+$ has only simple discontinuities: at each $t > 0$ the function is continuous from the right, and the left-limit $x^{\ominus}(t) = \lim_{s\uparrow\uparrow t} x(s)$ is well defined as a finite limit. By convention, $x^{\ominus}(0) = x(0)$. The ***jump function*** $\Delta x = x - x^{\ominus}$ gives the size of the jump at each point of $\mathbb{R}^+$, with $\Delta x(t) = 0$ at points of continuity.

The jump function can be nonzero for at most countably many times, for the following reason. For each $\epsilon > 0$ the set $J_\epsilon := \{t \in \mathbb{R}^+ : |\Delta x(t)| \ge \epsilon\}$ can put at most finitely points in each bounded interval: otherwise there would be a point t_0 with infinitely many members of J_ϵ in each of its neighborhoods, which would violate existence of left and right limits at t_0. Each J_ϵ is countable. All discontinuities of x must lie in $\cup_{k\in\mathbb{N}} J_{1/k}$, a countable set.

We could enumerate the locations of the jumps in x by first enumerating the points of J_1, then the points of $J_1 \backslash J_{1/2}$, then the points of $J_{1/2} \backslash J_{1/3}$, and so on. More formally, for each $k \in \mathbb{N}$, define $s_0^{(k)} = 0$ and

$$s_{i+1}^{(k)} = \min\{t \ge s_i^{(k)} : |\Delta x(t)| \ge 1\} \qquad \text{for } i \in \mathbb{N}.$$

Now suppose $\{X_t : t \in \mathbb{R}^+\}$ is a process with cadlag sample paths, adapted to a standard filtration. The jump process $\Delta X = X - X^{\ominus}$ picks off all the jumps in the sample paths. It is progressively measurable, being a difference of two progressively measurable processes. If we mimic the definition of $s_i^{(j)}$ we get a countable collection of random variables $\{\sigma_i^{(j)}\}$ that identify all the discontinuities in the sample paths of X. Each $\sigma_i^{(j)}$ is actually a stopping time: it is a debut of a progressively measure set $((\sigma_{i-1}^{(j)}, \infty)) \cap \{(j-1)^{-1} > |\Delta X| \geq j^{-1}\}$. (In fact, it is the smallest point in the set.) If we relabel the stopping times as a single sequence we get a simple representation for the jump process,

<17> $$\Delta X = \sum\nolimits_{k\in\mathbb{N}} \Delta X(\tau_k)[[\tau_k]],$$

for a countable family of stopping times $\{\tau_k\}$.

We could even assume that the graphs of the stopping times are disjoint: the random variable

$$\tau_i' := \begin{cases} \tau_i(\omega) & \text{if } \tau_i(\omega) \neq \tau_j(\omega) \text{ for } j < i \\ \infty & \text{otherwise} \end{cases}$$

has the progressively measurable set $[[\tau_i]] \backslash \cup_{j<i} [[\tau_j]]$ as its graph, and hence τ_i' is a stopping time. All (ω, t) that occur as points of $\cup_k[[\tau_k]]$ also occur as points of $\cup_k[[\tau_k']]$.

If X happens to be a predictable process, the jump process ΔX is a difference of two predictable processes. Each of the sets corresponding to the deterministic J_ϵ is a predictable set; each of the stopping times for the jumps is a predictable stopping time, by Example <5>. The size of the jump at each τ_k is $\Delta X(\tau_k)\{\tau_k < \infty\}$, which is $\mathcal{F}(\tau_k-)$-measurable, by Example <11>. This decomposition gives half of the characterization of predictable processes. The other half follows from Problems [2] and Problem [3].

<18> **Theorem.** *For a standard filtration, an adapted cadlag process is predictable if and only if there exists a countable collection of predictable stopping times $\{\tau_k\}$ such that*

(i) jumps of X can occur only at the $\{\tau_k\}$ times;

(ii) for each k, the jump size $\Delta X(\tau_k)\{\tau_k < \infty\}$ is $\mathcal{F}(\tau_k-)$-measurable.

10.7 Problems

[1] Let ξ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that supports a standard filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$. Let $\{X_t : 0 \leq t \leq \infty\}$ be a cadlag version of the uniformly integrable martingale $\mathbb{P}(\xi \mid \mathcal{F}_t)$.

(i) For each stopping time τ show that $X_\tau = \mathbb{P}(\xi \mid \mathcal{F}_\tau)$ almost surely.

(ii) If τ isa predictable stopping time, show that $X_\tau^{\ominus} = \mathbb{P}(\xi \mid \mathcal{F}(\tau-))$ almost surely.

[2] Suppose $H(\omega)$ is $\mathcal{F}(\tau-)$-measurable, for a predictable stopping time τ. Show that $H(\omega)[[\tau]]$ is predictable. Hint: Check the assertion for H equal to each of the generators of $\mathcal{F}(\tau-)$, by noting that $F_0[[\tau]] = (\mathbb{R}^+ \otimes F_0) \cap [[\tau]]$ and $F_s\{s < \tau\}[[\tau]] = (s, \infty) \otimes F_s \cap [[\tau]]$.

[3] Prove that a process satisfying conditions (i) and (ii) of Theorem <18> must be predictable. Hint: Use Problem [2] to show that each of the summands in the expression <17> for ΔX is predictable. Deduce that X equals the sum of a predictable process and an adapted left-continuous process.

10.8 Notes

I adapted most of the results for this Project from Chapter 1 of Métivier (1982), which gives a most readable account of a good chunk of the "Strasbourg theory of processes".

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