Project 11

Brownian Motion

Lect 22, Wednesday 8 April

11.1 Random elements of $C[0,\infty)$

Suppose $\{X_t : t \in \mathbb{R}^+\}$ is a process with continuous sample paths. That is, for each fixed ω the sample path $X(\omega,\cdot)$ is a member of $C[0,\infty)$, the set of all continuous real functions (not necessarily bounded) on $\mathbb{R}^+$. Equip $C[0,\infty)$ with its ***cylinder sigma-field*** $\mathcal{C}$, which is defined as the smallest sigma-field on $C[0,\infty)$ for which each coordinate map π_t is $\mathcal{C}\backslash\mathcal{B}(\mathbb{R})$-measurable. Then $\omega \mapsto X(\omega,\cdot)$ is an $\mathcal{F}\backslash\mathcal{C}$-measurable map from Ω into $C[0,\infty)$—see Problem [1]. Equip $C[0,\infty)$ with its metric for uniform convergence on compacta,

For fixed $t \in \mathbb{R}^+$, $\pi_t(x) := x(t)$ for $x \in C[0,\infty)$.

$$d(x,y) = \sum\nolimits_{k\in\mathbb{N}} 2^{-k} \min\left(1, \sup_{0\le t\le k} |x(t)-y(t)|\right)$$

(i) Show that d makes $C[0,\infty)$ a separable metric space.

(ii) Show that the Borel sigma-filed for d coincides with the cylinder sigma-firld $\mathcal{C}$.

The distribution of X is a probability measure defined on $\mathcal{C}$, the image of $\mathbb{P}$ under the map $\omega \mapsto X(\omega,\cdot)$. For example, for a standard Brownian motion, the distribution is called ***Wiener measure***, which I will denote by the symbol $\mathbb{W}$. In other words, if B is a standard Brownian motion, and at least if $f : C[0,\infty) \to \mathbb{R}^+$ is a $\mathcal{C}\backslash\mathcal{B}(\mathbb{R}^+)$-measurable function, then

Note: "at least" is an invitation for you to extend the result to a larger set of functions

$$\mathbb{P}^\omega f(X(\omega,\cdot)) = \mathbb{W}^x f(x).$$

Sometimes I will slip into old-fashioned terminology and call a real-valued (or extended-real-valued) function a ***functional*** if it is defined on a space of functions.

The stochastic process $\{X_t : t \in \mathbb{R}^+\}$ defines on Ω a ***natural filtration*** (sometimes called a raw filtration),

$$\mathcal{F}_t^\circ := \sigma\{X_s : 0 \le s \le t\} \qquad \text{for } t \in \mathbb{R}^+$$

with $\mathcal{F}_\infty^\circ := \sigma\{X_s : s \in \mathbb{R}^+\}$. Problem [2] shows that each $\mathcal{F}_\infty^\circ$-measurable random variable on Ω can be expressed as a composition $h(X(\omega,\cdot))$ with h

a $\mathcal{C}$-measurable functional on $C[0,\infty)$. Moreover, if for each fixed $\tau \in \mathbb{R}^+$ we define the stopping operator $K_\tau : C[0,\infty) \to C[0,\infty)$ by

$$(K_\tau x)(t) = x(\tau \wedge t) \qquad \text{for } t \in \mathbb{R}^+,$$

then each $\mathcal{F}_t^\circ$-measurable random variable on Ω can be expressed as a composition $h(K_t X(\omega,\cdot))$ with h a $\mathcal{C}$-measurable functional on $C[0,\infty)$.

11.2 Decomposition of Brownian motion sample paths

Sometimes it is useful to have the underlying filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ built into the definition of the Brownian motion.

<1> **Definition.** *A process $\{B_t : t \in \mathbb{R}^+\}$ with continuous sample paths is called a Brownian motion with respect to a filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$ if*

(i) B is adapted to the filtration

(ii) for each $s < t$, the increment $B_t - B_s$ is $N(0, t-s)$ distributed and is independent of $\mathcal{F}_s$

Remark. It is not necessary to add the further requirement that the B process should have independent increments (as in the definition given in Chapter 1) because it follows directly from (ii).

Not surprisingly, a Brownian motion B in the sense of Chapter 1 is also a Brownian motion with respect to its natural filtration. As you will see in the next Section, it is useful that B is also a Brownian motion, in the sense of Definition <1>, with respect to a slightly larger, standard filtration.

For a fixed $\tau \in \mathbb{R}^+$, a generating class argument would show that the restarted process $R_\tau B$, defined by

$$R_\tau B(t) := B(\tau + t) - B(\tau) \qquad \text{for } t \in \mathbb{R}^+,$$

is also a Brownian motion, which is independent of $\mathcal{F}_\tau$. This property has a few useful consequences. Define the shift operator S_τ by

$$(S_\tau x)(t) = \begin{cases} 0 & \text{for } 0 \le t < \tau \\ x(t-\tau) & \text{for } t \ge \tau \end{cases}$$

Then:

(i) B has the same distribution as $K_\tau B + S_\tau \widetilde{B}$, where $\widetilde{B}$ is a new standard Brownian motion that is independent of B.

(ii) At least for each $\mathcal{C}$-measurable functional $h : C[0,\infty) \to \mathbb{R}^+$,

$$\mathbb{P}\left(h(B) \mid \mathcal{F}_\tau\right) = \mathbb{W}^x h(K_\tau B + S_\tau x) \qquad \text{almost surely.}$$

That is, for each $X \in \mathcal{M}^+(\Omega, \mathcal{F}_\tau)$ and each h as in (ii),

$$\mathbb{P}Xh(B) = \mathbb{P}^\omega\left(X(\omega)\mathbb{W}^x h(K_\tau B(\omega,\cdot) + S_\tau x)\right)$$

(iii) At least for each $\mathcal{B}(\mathbb{R}) \otimes \mathcal{C}$-measurable map $f : \mathbb{R} \times C[0,\infty) \to \mathbb{R}^+$, and each $\mathcal{F}_\tau$-measurable random variable Y,

$$\mathbb{P}f(Y,B) = \mathbb{P}^\omega \mathbb{W}^x f(Y, K_\tau B + S_\tau x)$$

See Pollard (2001, Chapter 9) for extensions of the these assertions—the so-called strong Markov property for Brownian motion—when τ is a stopping time.

11.3 The Brownian filtration

Let $\{\mathcal{F}_t^\circ : t \in \mathbb{R}^+\}$ be the natural filtration for a Brownian motion B (with continuous paths) defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Complete this filtration by adding the set $\mathcal{N}$ of all $\mathbb{P}$-negligible subsets of Ω to the generating class,

$$\mathcal{F}_t = \sigma\left(\mathcal{F}_t^\circ \cup \mathcal{N}\right).$$

<2> **Theorem.** *The process B is also a Brownian motion with respect to the completed filtration, which is standard.*

PROOF The first assertion is easy to prove. Suppose $s < t$ and $F \in \mathcal{F}_s$.

(i) Explain why there exists a set F^* in $\mathcal{F}_s^\circ$ for which $\mathbb{P}|F - F^*| = 0$.

(ii) Show that

$$\mathbb{P}F^* f\left(B_t - B_s\right) = \left(\mathbb{P}F^*\right)\left(\mathbb{P}f(Z)\right) \qquad \text{where } Z \sim N(0, t-s)$$

at least for each bounded $\mathcal{B}(\mathbb{R})$-measurable funtion f.

The standard property of the completed filtration requires a little more work. Consider an F in $\mathcal{F}_{s+}$.

(iii) Let $s_n = s + n^{-1}$. Explain why there exist sets $F_n \in \mathcal{F}_{s_n}$ and $N_n \in \mathcal{N}$ such that $|F - F_n| = N_n$ for every n.

(iv) Let $F^* = \limsup_{n\to\infty} F_n$. Show $|F - F^*| \le \cup_n N_n \in \mathcal{N}$ and $F^* \in \mathcal{F}^\circ_{s+}$.

(v) Suppose h is a bounded continuous functional and k is a bounded $\mathcal{C}$-measurable functional, both defined on $\mathcal{X} := C[0,\infty)$. Show that

$$\mathbb{P}k(K_sB)F^*h(R_{s_n}B) = \mathbb{P}k(K_sB)F^*\mathbb{W}h \qquad \text{for each } n.$$

(vi) Let $X := K_sB$ and $Y := R_sB$. Note that X and Y are independent. Let n tend to infinity in the previous step to deduce that $\mathbb{P}k(X)F^*h(Y) = \mathbb{P}k(X)F^*\mathbb{P}h(Y)$.

(vii) Explain why $F^* = g(X,Y)$ for some bounded, product measurable functional on $\mathcal{X} \times \mathcal{X}$.

(viii) Define $G(x) := \mathbb{W}^y g(x,y)$. Explain why $\mathbb{P}k(X)F^* = \mathbb{P}k(X)G(X)$.

(ix) Deduce that $\mathbb{P}k(X)\left(F^* - G(X)\right)h(Y) = 0$.

(x) Use a generating class argument to show the $k(X)h(Y)$ in the previous equality can be replaced by $f(X,Y)$, for a general bounded, product measurable f.

(xi) By an appropriate choice of f, deduce that $F \in \mathcal{F}_s$.

□

From now on I will refer to the completed natural filtration as the ***Brownian filtration***. processes adapted to his filtration is not only standard, it also force Some important processes—local martingales—that are adapted to the Brownian filtration have two rather surprising properties: they have continuous sample paths and they can be represented as stochastic integrals. These two results are proved in the next Section.

11.4 Surprising properties for local martingales

Lect 23, Monday 12 April

<3> **Definition.** *A cadlag process $\{M_t : t \in \mathbb{R}^+\}$ is said to be a local martingale if there exist stopping times $\{\tau_k : k \in \mathbb{N}\}$ with $\sup_k \tau_k = \infty$ for which each $M_{\wedge\tau_k} - M_0$ is a martingale.*

<4> **Theorem.** *Every local martingale with respect to the Brownian filtration has almost all of its sample paths continuous. Consequently, every local martingale is also a locally square-integrable martingale.*

Proof Without loss of generality, assume the local martingale M starts with $M_0 \equiv 0$.

First consider the case of a cadlag martingale $\{X_t : t \in \mathbb{R}^+\}$ with respect to the Brownian filtration. It is enough to show that almost all sample paths are continuous on each bounded interval $[0, k]$. For simplicity suppose $k = 1$.

(i) Explain why X_1 can be expressed as $f(B)$ for some $\mathcal{C}\backslash\mathcal{B}(\mathbb{R})$-measurable, $\mathbb{W}$-integrable functional f on $C[0, \infty)$.

(ii) If f is bounded and continuous (for the uccp metric), show that $t \mapsto f(K_t B + S_t y)$ is continuous, for each fixed y in $C[0, \infty)$. Use the representation $X_t = \mathbb{P}_t X_1 = \mathbb{W}^y f(K_t B + S_t y)$ almost surely, for each fixed t in $[0, 1]$, to show that $X(\omega, \cdot)$ is continuous on $[0, 1]$ for almost all ω.

(iii) For the general f, show that there exists a sequence of bounded, continuous functionals $\{f_n\}$ for which $\mathbb{W}|f - f_n| \leq 4^{-n}$. Hint: This approximation is a consequence of general properties of Borel probability measures defined on metric spaces.

(iv) Let X_n be a version of the martingale $\mathbb{P}_t f_n(B)$, for $0 \leq t \leq 1$, with continuous sample paths. Show that $\{|X_n(t) - X(t)| : 0 \leq t \leq 1\}$ is a uniformly integrable submartingale with cadlag sample paths.

(v) Define stopping times $\tau_n := 1 \wedge \min\{t : |X_n(t) - X(t)| \geq 2^{-n}\}$. Show that

$$
\begin{aligned}
\mathbb{P}\{\sup_{0\leq t\leq 1} |X_n(t) - X(t)| > 2^{-n}\} &\leq 2^n \mathbb{P}|X_n(\tau_n) - X(\tau_n)| \\
&\leq 2^n \mathbb{P}|X_n(1) - X(1)| \\
&= 2^n \mathbb{P}|f_n(B) - f(B)| \\
&\leq 2^{-n}
\end{aligned}
$$

(vi) Deduce that $\sum_n \mathbb{P}\{\sup_{0\leq t\leq 1} |X_n(t) - X(t)| > 2^{-n}\} < \infty$ and hence $\sup_{0\leq t\leq 1} |X_n(t) - X(t)| \to 0$ almost surely.

(vii) Conclude that almost all sample paths of X are continuous.

(viii) Extend the argument to the case of a local martingale. Hint: If $M_{\wedge\tau_k}$ has (almost all) continuous paths on $[0,1]$ for each k, and if $\tau_k \uparrow \infty$, what do you know about almost all paths of M?

□

The other surprising property is little more than an application of the Itô formula. To simplify the argument, work on a bounded interval. Remember that the submartingale $\{B_t^2 : 0 \le t \le 1\}$ has Doléans measure $\mu = \mathbb{P} \otimes \mathfrak{m}$, where $\mathfrak{m}$ denotes Lebesgue mesure on $[0,1]$.

Lect 24, Wednesday 14 April

<*5*> **Theorem.** *For each* $X \in \mathcal{L}^2(\Omega, \mathcal{F}_1, \mathbb{P})$, *where* $\{\mathcal{F}_t : 0 \le t \le 1\}$ *is the Brownian filtration, there exists an* $H \in \mathcal{L}^2(\mu)$ *such that* $X - \mathbb{P}X = H \bullet B_1$ *almost surely.*

Proof (Sketch) Without loss of generality, suppose $\mathbb{P}X = 0$.

(i) Explain why $\mathcal{R} := \{H \bullet B_1 : H \in L^2(\mu)\}$ is a closed vector subspace of $\mathcal{L}^2(\mathbb{P}, \mathcal{F}_1)$.

(ii) Let Z denote the component of X that is orthogonal to $\mathcal{R}$. That is, $X = Z + K \bullet B_1$ for some $K \in \mathcal{L}^2(\mu)$ and $\mathbb{P}Z(H \bullet B)_1 = 0$ for all H in $\mathcal{L}^2(\mu)$. Show that $\mathbb{P}Z = 0$.

(iii) Explain why it is enough to prove $Z = 0$ almost surely.

(iv) Explain why it suffices to show $\mathbb{P}Zf(B) = 0$ for all bounded, $\mathcal{C}$-measurable functionals f on $C[0,1]$.

(v) Explain why it suffices to consider functionals f that depend on B only through its increments $Y_j = B_{t_{j+1}} - B_{t_j}$ for a fixed set of times $0 = t_0 < t_1 < \cdots < t_k = 1$. That is, why is it enough to prove $\mathbb{P}Zg(\mathbf{Y}) = 0$ for all bounded, measurable functions g on $\mathbb{R}^k$?

(vi) Invoke Problem [6] to show that it is enough to prove $\mathbb{P}Z\exp(i\theta'\mathbf{Y}) = 0$ for all θ in $\mathbb{R}^k$.

(vii) Work with stochastic integral notation. Show that $\theta'\mathbf{Y} = H \bullet B_1$, where $H := \sum_{j=0}^{k-1} \theta_j ((t_j, t_{j+1}]]$.

(viii) Show that $H \bullet B$ has a deterministic quadratic variation process, $A_t := [H \bullet B, H \bullet B]_t = \int_0^t H^2(s)\,ds$.

(ix) Use Itô to show that

$$W_1 = 1 + i(WH) \bullet B_1 \qquad \text{where } W_t := \exp(iH \bullet B_t + \tfrac{1}{2}A_t).$$

(x) Deduce that

$$\exp(A_1/2)\mathbb{P}Z\exp(i\theta \cdot \mathbf{Y}) = 0.$$

(xi) Are we done?

□

<6> **Corollary.** *For each local martingale M adapted to the Brownian filtration there exists an H in $\text{loc}\mathcal{L}^2(\mu)$ such that $M_t = M_0 + H \bullet B_t$ for $0 \le t \le 1$.*

Proof Without loss of generality, suppose $M_0 = 0$. Define stopping times $\tau_k := 1 \wedge \inf\{t : |M_t| \ge k\}$.

(i) Why does $M_{\wedge\tau_k}$ belong to $\mathcal{M}_0^2[0,1]$?

(ii) For each k, explain why there exists an $H_k \in \mathcal{L}_2(\mu)$ such that

$$M_{t\wedge\tau_k} = \big(H_k((0,\tau_k]]\big) \bullet B_t \qquad \text{for } 0 \le t \le 1.$$

(iii) Deduce that $(H_k((0,\tau_k]]) \bullet B_1 = (H_{k+1}((0,\tau_k]] \bullet B_1$ almost surely.

(iv) Deduce that $H_k((0,\tau_k]] - H_{k+1}((0,\tau_k]] = 0$ almost everywhere $[\mu]$.

Am I just repeating the construction for the $\text{loc}\mathcal{M}_0^2[0,1]$ stochastic integral?

(v) Show that the H_k processes can be pasted together to create an H in $\text{loc}\mathcal{L}^2(\mu)$ for which $M_t = H \bullet B_t$ almost surely.

□

Remark. Should I extend to general $\mathcal{F}_1$-measurable random variables, perhaps using the method of Dudley (1977), getting a representation $Y_0 + H \bullet B_1$ with $H \in \text{loc}\mathcal{H}_{\text{Bdd}}$.

11.5 Change of measure for Brownian motion

Let $\{B_t : 0 \le t \le 1\}$ be a Brownian motion with respect to a standard filtration $\{\mathcal{F}_t : 0 \le t \le 1\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. Write $\mathcal{U}$ for its quadratic variation process, $\mathcal{U}_t = t$.

For each $\alpha \in \mathbb{R}$, the process

$$q_t = \exp\left(\alpha B_t - \tfrac{1}{2}\alpha^2 t\right) \qquad \text{for } 0 \le t \le 1$$

is a nonnegative martingale, with $\mathbb{P}q_t = \mathbb{P}q_0 = 1$. Define a new probability measure $\mathbb{Q}_\alpha$ on $\mathcal{F}_1$ by specifying q_1 to be its density with respect to $\mathbb{P}$. That is, $\mathbb{Q}_\alpha X = \mathbb{P}(Xq_1)$, at least for all bounded random variables X.

(i) Show that $\mathbb{Q}_\alpha$ is equivalent to $\mathbb{P}$, in the sense that both measures have the same collection $\mathcal{N}$ of negligible sets.

(ii) Show that $\mathbb{Q}_\alpha X = \mathbb{P}(Xq_t)$ if X is $\mathcal{F}_t$-measurable. Explain why q_t is a Radon-Nikodym density for $\mathbb{Q}_\alpha$ with respect to $\mathbb{P}$ when both measures are restricted to $\mathcal{F}_t$.

(iii) Define $B_t^* := B_t - \alpha t$. For fixed s and $t = s + \delta$, define $Y_1 := B_s$, $Y_2 := B_t - B_s$, and $Y_3 := B_1 - B_t$. For a fixed F in $\mathcal{F}_s$ and $\theta \in \mathbb{R}$, show that

$$\begin{aligned}
&\mathbb{Q}_\alpha F \exp\left(i\theta(Y_2 - \alpha\delta)\right) \\
&\qquad = \mathbb{P}(F q_s \exp\left((i\theta + \alpha)Y_2 - i\alpha\theta\delta + \alpha Y_3 - \tfrac{1}{2}\alpha^2(1-s)\right) \\
&\qquad = \mathbb{Q}_\alpha F \exp\left(-\tfrac{1}{2}\theta^2\delta\right)
\end{aligned}$$

(iv) Deduce that, under $\mathbb{Q}_\alpha$, the B^* process is a standard Brownian motion.

11.6 The Black-Scholes formula

Stock prices (in units so that $S_0 \equiv 1$) are sometimes modeled by a continuous process driven by a Brownian motion, B, on $[0, 1]$;

<7> $$S_t = \exp((\mu - \tfrac{1}{2}\sigma^2)t + \sigma B_t) \qquad \text{for } 0 \le t \le 1$$

for constants $\sigma > 0$ (assumed known) and μ (unknown). That is,

$$\begin{aligned}
S_t &= \psi_{\mu,\sigma}(B_t, \mathcal{U}_t) \qquad \text{where } \psi_{\mu,\sigma}(x,y) := \exp(\sigma x + (\mu - \tfrac{1}{2}\sigma^2)y) \\
&= 1 + \sigma S \bullet B_t + \mu S \bullet \mathcal{U}_t \qquad \text{by the Itô formula.}
\end{aligned}$$

In more traditional notation,

$$dS_t = \sigma S_t \, dB_t + \mu S_t \, dt, \qquad \text{or} \qquad \frac{dS_t}{S_t} = \sigma \, dB_t + \mu \, dt.$$

Roughly speaking, the relative increments of S behave like the increments of a Brownian motion with drift μ.

> **Remark.** Throughout this Section I will ignore inflation. Equivalently, I could express the value of stock as a multiple of a bond price.

Question: Suppose $X = f(S)$, with f a $\mathcal{C}$-measurable functional on $C[0,1]$. How much should one pay at time 0 in order to receive the amount X at time 1? Call this fair price C_0, a constant that is declared at time 0.

To answer the question I will first assume that μ is zero, so that

<8> $$S_t = \psi_{0,\sigma}(B_t, \mathcal{U}_t) = 1 + \sigma S \bullet B_t \qquad \text{for } 0 \le t \le 1.$$

A change-of-measure will later restore the μ. I will also assume $\mathbb{P}X^2 < \infty$.

Think of $\psi_{0,\sigma}$ as defining a continuous map Ψ_σ from $C[0,1]$ to $C[0,1]$,

$$\Psi_\sigma(x)(t) = \exp\left(\sigma x(t) - \tfrac{1}{2}\sigma^2 t\right) \qquad \text{for } 0 \le t \le 1.$$

Then $X = f(\Psi_\sigma B)$, a $\mathcal{C}$-measurable functional of the whole Brownian motion path. By Theorem <5>, there exists a predictable process H, which is square-integrable with respect to the Doléans measure, such that

<9> $$X = c + H \bullet B_1 \qquad \text{where } c := \mathbb{P}X.$$

The process $(\sigma S)^{-1}$ is locally bounded. If we integrate this process with respect to the semimartingale S we get, via <8>,

$$B = \frac{1}{\sigma S} \bullet S \qquad \text{and} \qquad X = c + K \bullet S_1 \qquad \text{where } K_t := H_t/(\sigma S_t)$$

Temporarily suppose K is of the form $\sum_{j \le k} h_j(\omega)((\tau_j, \tau_j']]$ for stopping times $\tau_j \le \tau_j'$ and $\mathcal{F}(\tau_j)$-measurable random variables $h_j(\omega)$. Engage in the following trades:

a) At time zero pay C_0 in order to receive the random amount X at time 1.

b) For $j = 1, 2, \ldots, k$, pay h_j at time τ_j, at price $S(\tau_j)$ per share, then sell the same shares at time τ_j' at price $S(\tau_j')$ per share.

The total profit at time 1 from the trading will be

$$X - C_0 + \sum\nolimits_{j \le k} h_j \left(S(\tau_j') - S(\tau_j) \right) = X - C_0 - K \bullet S_1 = c - C_0.$$

If $c > C_0$ we would be very happy: pure profit without risk. If $c < C_0$, take the other side of each trade to make a profit of $C_0 - c$ without risk. The set of trades in either case would be called an ***arbitrage*** scheme. Only if $C_0 = c$ does the opportunity to make a riskless profit by arbitrage disappear. The fair price for X at time zero should be $c = \mathbb{P}X$.

Lect 25, Monday 19 April

It is traditional to interpret more general K processes as defining a limit of a sequence of discrete trades, or an idealized scheme for continuous trading, that would again force $C_0 = \mathbb{P}X$ as the fair price.

For the arbitrage threat to be credible, the trading scheme K needs to be determined explicitly. I know how to find K if X is of the form $X = g(L_1)$ where $L_t := \log S_t = \sigma B_t - \frac{1}{2}\sigma^2$. First note that

$$L_1 = L_t + \sigma(B_1 - B_t) - \tfrac{1}{2}(1-t) = L_t + \gamma Z - \tfrac{1}{2}\gamma^2$$

where $\gamma := \sigma\sqrt{1-t}$ and $Z := (B_1 - B_t)/\sqrt{1-t} \sim N(0,1)$ if $t > 0$. Then the martingale $X_t := \mathbb{P}_t X$ has the representation

$$X_t = G(L_t, \sigma\sqrt{1-t}) = G(\sigma B_t - \tfrac{1}{2}\sigma^2 t, \sigma\sqrt{1-t})$$

where

$$\begin{aligned} G(x,\gamma) &= \mathbb{P}g(x + \gamma Z - \tfrac{1}{2}\gamma^2) \\ &= \int_{-\infty}^{\infty} g(x + \gamma z - \tfrac{1}{2}\gamma^2)\phi(z)\,dz \qquad \text{with } \phi(z) = \frac{\exp\left(-\frac{1}{2}z^2\right)}{\sqrt{2\pi}} \\ &= \gamma^{-1}\int_{-\infty}^{\infty} g(y)\phi\left(\frac{y - x - \frac{1}{2}\gamma^2}{\gamma}\right)dy \qquad \text{for } \gamma > 0 \end{aligned}$$

and $G(x,0) = g(x)$.

The final integral representation for G shows that X_t is a very smooth function of B_t and $\mathfrak{U}_t$ for all $0 \le t < 1$. Apply the Itô formula.

$$X_t = C_0 + \sigma G_x \bullet B_t + W \bullet \mathfrak{U}_t \qquad \text{for } 0 \le t < 1,$$

where W is some locally bounded, predictable process involving G_{xx} and G_t. We don't need to calculate W explicitly, because the fact that $X_t - C_0 - \sigma G_x \bullet B_t$ is a local martingale forces $W \bullet \mathfrak{U} = 0$ (Problem [5]). In particular, $H = \sigma G_x$ is the predictable process for which $X = C_0 + H \bullet B_t$ and $C_0 = G(0,\sigma)$.

Remark. Does it matter that we might have trouble defining H_1? Does continuity of the sample paths of X take care of the problem?

You might try to explain why X_t is the fair price to pay at time t in order to receive X at time 1.

Now consider the case where μ is unknown, possibly nonzero. Let $\mathbb{Q}$ be the probability measure with density $\exp(\alpha B_1 - \frac{1}{2}\alpha^2)$ with respect to $\mathbb{P}$, where $\alpha = \mu/\sigma$. From Section 5 we know that $B_t^* = B_t - \alpha t$ is a Brownian motion under $\mathbb{Q}$. Also the process

$$S_t = \exp\left(\sigma B_t - \tfrac{1}{2}\sigma^2 t\right) = \exp\left(\sigma B_t^* + (\mu - \tfrac{1}{2}\sigma^2)t\right)$$

is a $\mathbb{Q}$-semimartingale with the distribution specified by <7>

The characterization of the stochastic integral in Theorem 8.5 should convince you that the representation $X = C_0 + K \bullet S_1$ derived under $\mathbb{P}$ is also valid under $\mathbb{Q}$. (By mutual absolute continuity of $\mathbb{P}$ and $\mathbb{Q}$, uccp means the same thing under both measures.) We still have a trading strategy K, *which does not depend on the unknown* μ, to enforce C_0 as the fair price for X.

<10> **Example.** A European option gives the buyer the right to pay an amount K at time 1 in order to receive one unit of stock, regardless of the stock price at that time. If $S_1 \le K$ the option will be worthless; if $S_1 > K$ it returns a profit of $S_1 - K$. In short, the option returns $X = (S_1 - K)^+$ at time 1. That is, it corresponds to the function $g(x) = (e^x - K)^+$.

Calculate. Note that

$$\exp\left(x + \gamma z - \tfrac{1}{2}\gamma^2\right) \ge K \qquad \text{iff} \quad z \ge r := \left(\log K - x + \tfrac{1}{2}\gamma^2\right)/\gamma.$$

For $\gamma > 0$ we then get

$$\begin{aligned} G(x,\gamma) &= \int_{-\infty}^{\infty} g(x + \gamma z - \tfrac{1}{2}\gamma^2)\phi(z)\,dz \\ &= \int_r^{\infty} \exp\left(x + \gamma z - \tfrac{1}{2}\gamma^2\right)\phi(z)\,dz - K\int_r^{\infty} \phi(z)\,dz \\ &= e^x \int_r^{\infty} \phi(z-\gamma)\,dz - K\int_r^{\infty}\phi(z)\,dz \\ &= e^x\overline{\Phi}(r-\gamma) - K\overline{\Phi}(r) \end{aligned}$$

where $\overline{\Phi}(t) := \mathbb{P}\{Z \ge t\}$ for $Z \sim N(0,1)$. In particular,

$$C_0 = G(0,\sigma) = \overline{\Phi}\left(\frac{\log K - \frac{1}{2}\sigma^2}{\sigma}\right) - K\overline{\Phi}\left(\frac{\log K + \frac{1}{2}\sigma^2}{\sigma}\right)$$

This result will look more like the traditional Black-Scholes formula displayed by Steele (2001, Section 10.3) if you put his initial share price S equal to 1, take his interest rate r equal to zero, and use the fact that $\overline{\Phi}(y) - \Phi(-y)$.

□

11.7 Problems

[1] Let ψ be a map from $(\Omega, \mathcal{F})$ to $C[0,\infty)$.

(i) Show that ψ is $\mathcal{F}\backslash\mathcal{C}$-measurable if and only if $\pi_t \circ \psi$ is $\mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurable for each $t \in \mathbb{R}^+$.

(ii) Deduce that a stochastic process $\{X_t : t \in \mathbb{R}^+\}$ with continuous sample paths defines an $\mathcal{F}\backslash\mathcal{C}$-measurable map from Ω into $C[0,\infty)$.

[2] Suppose X is a stochastic process with sample paths in $C[0,\infty)$. For each fixed t, define $\mathcal{F}_t^\circ := \sigma\{X_s : 0 \le s \le t\}$.

(i) Show that $\mathcal{F}_t^\circ$ is the smallest sigma-field for which the map $\omega \mapsto K_t X(\omega, \cdot)$ is $\mathcal{F}_t^\circ\backslash\mathcal{C}$-measurable.

(ii) Deduce (cf. Pollard 2001, Problem 2.3) that each $\mathcal{F}_t^\circ$-measurable random variable can be factorized as $h(K_t X(\omega,\cdot))$ for some $\mathcal{C}$-measurable functional $h : C[0,\infty) \to \overline{\mathbb{R}}$.

[3] Suppose $Z \in \mathbb{FV}_0 \cap \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ and Z has continuous sample paths. Show that $Z_t = 0$ almost surely, for each t. Hint: Use the fact that $[Z,Z] = 0$ to deduce that $Z^2 = 2Z \bullet Z \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$. Find a sequence of stopping times $\tau_k \uparrow \infty$ for which $\mathbb{P}Z_{t\wedge\tau_k}^2 = 0$ for each t.

[4] Suppose $M \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$ has continuous sample paths. Suppose $A \in \mathbb{FV}_0$ also has continuous paths and $M^2 - A \in \text{loc}\mathcal{M}_0^2(\mathbb{R}^+)$. Deduce that $A = [M,M]$. Hint: Apply Problem [3] to $[M,M] - A$.

[5] Suppose $W \in \text{loc}\mathcal{H}_{\text{Bdd}}$ is such that $M_t := W \bullet \mathcal{U}_t$ is a local martingale. Use the fact that $M^2 - [M,M]$ is a local martingale to deduce that M is indistinguishable from the zero process. Hint: $[M,M] = W^2 \bullet [\mathcal{U},\mathcal{U}] = 0$.

[6] Let X be an integrable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $Y = (Y_1, \dots, Y_k)$ be a vector of random variables such that $\mathbb{P}X \exp(i\theta' Y) = 0$ for all $\theta = (\theta_1, \dots, \theta_k)$ in $\mathbb{R}^k$. Show that $\mathbb{P}(Xg(Y)) = 0$ for all bounded, measurable g by these steps.

(i) Let ϕ_σ denote the $N(0, \sigma^2 I_k)$ density. Show (cf. Pollard 2001, Section 8.2) that

$$\phi_\sigma(t) = (2\pi)^{-k} \mathfrak{m}^\theta \exp\left(-i\theta' t - \tfrac{1}{2}\sigma^2 |\theta|^2\right)$$

where $\mathfrak{m}$ denotes k-dimensional Lebesgue measure.

(ii) Suppose $Z \sim N(0, \sigma^2 I_k)$ independently of (X, Y). For each continuous, real-valued h with compact support on $\mathbb{R}^k$, justify all the implicit appeals to Fubini to show that

$$\begin{aligned}
\mathbb{P} X h(Y+Z) &= \mathbb{P}^\omega X(\omega) \mathfrak{m}^z h\left(Y(\omega) + z\right) \phi_\sigma\left(z\right) \\
&= \mathbb{P}^\omega X(\omega) \mathfrak{m}^u h(u) \phi_\sigma\left(u - Y(\omega)\right) \\
&= \mathfrak{m}^u \mathfrak{m}^\theta h(u) (2\pi)^{-k} \mathbb{P} X \exp\left(-i\theta'(u - Y) - \tfrac{1}{2}\sigma^2|\theta|^2\right) \\
&= 0
\end{aligned}$$

(iii) Let σ tend to zero to deduce that $\mathbb{P} X h(Y) = 0$.

(iv) Use a lambda-space argument to extend to all bounded, measurable g.

[7] Let X and Y be independent Brownian Motions.

(i) Show that both $(X+Y)/\sqrt{2}$ and $(X-Y)/\sqrt{2}$ are also Brownian Motions.

(ii) Deduce that $[X, Y] = 0$.

The next problem presents the standard example of a uniformly integrable local martingale that is not of class [D].

[8] Let $\boldsymbol{B} = (1+X, Y, Z)$ be a three-dimensional Brownian Motion started from $\boldsymbol{u} = (1, 0, 0)$. (The three processes X, Y, and Z are independent Brownian Motions started from zero.) Write $\mathbf{x}$ for (x, y, z). Define $f(\mathbf{x}) = 1/\|\mathbf{x}\|$ on $\mathbb{R}^3 \backslash \{0\}$. Define a process $M(t) = 1/\|\boldsymbol{B}(t)\|$. Better to start at origin, and work with distance to $\boldsymbol{u}$?

(i) Use the Multiprocess Itô Formula to show that $M \in \mathrm{loc}\mathcal{M}^2(\mathbb{R}^+)$. Hint: Show that on the open region $\mathbb{R}^3 \backslash \{0\}$ the function f is harmonic:

$$\frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z} = 0.$$

(ii) Deduce that M is a positive supermartingale.

(iii) Let $\tau_k = \inf\{t : \|\boldsymbol{B}(t)\| \le 1/k\}$. Show that $M_{\wedge \tau_k} \in \mathcal{M}^2(\mathbb{R}^+)$.

(iv) Show that $C_0 := \int \{\|\mathbf{x}\| \le \frac{1}{2}\} \|\mathbf{x}\|^{-2} \, d\mathbf{x} < \infty$.

(v) Show that $\mathbb{P} M(t)^2 \le C_0 \exp(-(8t)^{-1}) t^{-3/2} + \mathbb{P}\left(8 \wedge \|\boldsymbol{B}(t)\|^{-2}\right)$.

(vi) Show that $\|\boldsymbol{B}(t)\|^2 \xrightarrow{\mathbb{P}} \infty$ as $t \to \infty$.

(vii) Deduce that $\sup_t \mathbb{P} M(t)^2 < \infty$ and $\mathbb{P} M(t) \to 0$ as $t \to \infty$.

(viii) Deduce that M is not a martingale, and hence M is not in class [D].

11.8 Notes

See Steele (2001, Chapter 10) and Chung and Williams (1990, Section 10.5) for slightly different ways to derive option prices by arbitrage arguments. I learned about the significance of semimartingales for option pricing from Harrison and Pliska (1981).

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