Project 4

Martingales, standard filtrations, and stopping times

Throughout this Project the index set T is taken to equal $\mathbb{R}^+$, unless explicitly noted otherwise.
Some things you might want to explain in your notebook:

(i) standard filtrations: Why are they convenient?

(ii) stopping times and related sigma-fields

(iii) How does progressive measurability help?

(iv) Is the (sub)martingale property preserved at stopping times?

(v) Cadlag versions of martingales.

You might want to explain any item in an enumerated list or prefaced by a bullet (•) symbol.

Lect 7, Monday 1 Feb

4.1 Filtrations

Start with a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $\mathcal{N}$ to consists of all sets $A \subseteq \Omega$ for which there exists some $F \in \mathcal{F}$ with $A \subseteq F$ and $\mathbb{P}F = 0$. The probability space (or $\mathbb{P}$ itself) is said to be ***complete*** if $\mathcal{N} \subseteq \mathcal{F}$.

- The probability measure $\mathbb{P}$ has a unique extension $\widetilde{\mathbb{P}}$ to a complete probability measure on $\widetilde{\mathcal{F}} = \sigma\{\mathcal{F} \cup \mathcal{N}\}$. In fact $\widetilde{\mathcal{F}}$ consists of all sets B for which there exist $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq B \subseteq F_2$ and $\mathbb{P}(F_2 \backslash F_1) = 0$. Necessarily, $\widetilde{\mathbb{P}}B = \mathbb{P}F_1$.

See, measure theory is fun.

A filtration on Ω is a family $\{\mathcal{F}_t : t \in T\}$ of sub-sigma-fields of $\mathcal{F}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s < t$. Define $\mathcal{F}_\infty := \sigma\left(\cup_{t \in T} \mathcal{F}_t\right)$. [In class I denoted this sigma-field by $\mathcal{F}_{\infty-}$, but that now seems a bit too fancy to me.]

A filtration is said to be ***right-continuous*** if $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t} \mathcal{F}_s$ for each t in T. A filtration $\{\mathcal{F}_t : t \in T\}$ is said to be ***standard*** if it is right continuous and if $\mathcal{N} \subseteq \mathcal{F}_0$.

- The filtration defined by $\mathcal{G}_t = \cap_{s>t}\sigma(\mathcal{F}_s \cup \mathcal{N})$ is standard.

The first big question is: Why worry about standard filtrations? I hope the following sections will give you some answers to this question.

4.2 Stopping times

A function $\tau : \Omega \to \overline{T} := T \cup \{\infty\}$ such that $\{\omega : \tau(\omega) \le t\} \in \mathcal{F}_t$ for each $t \in T$ is called a ***stopping time for the filtration***. For a stopping time τ define

$$\mathcal{F}_\tau = \{F \in \mathcal{F}_\infty : F\{\tau \le t\} \in \mathcal{F}_t \text{ for each } t \in T\}$$

(i) If the filtration is right continuous and if $\{\tau < t\} \in \mathcal{F}_t$ for each $t \in T$ then τ is a stopping time.

(ii) Show that $\mathcal{F}_\tau$ is a sigma-field.

(iii) Show that τ is $\mathcal{F}_\infty$-measurable.

(iv) Show that an $\mathcal{F}_\infty$-measurable random variable Z is $\mathcal{F}_\tau$-measurable if and only if $Z\{\tau \le t\}$ is $\mathcal{F}_t$-measurable for each $t \in T$.

4.3 Progressive measurability

If $\{X_t : t \in T\}$ is adapted and τ is a stopping time, when is the function

$$\omega \mapsto X(\tau(\omega), \omega)\{\tau(\omega) < \infty\}$$

$\mathcal{F}\tau$-measurable? A sufficient condition is that X is ***progressively measurable***, that is, the restriction of X to $[0, t] \times \Omega$ is $\mathcal{B}_t \otimes \mathcal{F}_t$-measurable for each $t \in T$.

Abbreviate $\mathcal{B}([0, t])$, the Borel sigma-field on $[0, t]$, to $\mathcal{B}_t$.

(i) [Warmup] Suppose τ takes values in T and is $\mathcal{F}$ measurable. If X is $\mathcal{B}(T) \otimes \mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurable, show $X(\tau(\omega), \omega)$ is $\mathcal{F}$-measurable.

$$\begin{array}{lccccc} & \omega & \overset{\psi}{\mapsto} & (\tau(\omega), \omega) & \overset{X}{\mapsto} & X(\tau(\omega), \omega) \\ & \in \Omega & & \in\ T \times \Omega & & \in \mathbb{R} \\ \sigma\text{-field:} & \mathcal{F} & & \mathcal{B}(T) \otimes \mathcal{F} & & \mathcal{B}(\mathbb{R}) \end{array}$$

Show that ψ is $\mathcal{F}\backslash\mathcal{B}(T)\otimes\mathcal{F}$-measurable by a generating class argument, starting from $\psi^{-1}[0, t] \times F$.

(ii) Now suppose X is progressively measurable and τ is a stopping time. For a fixed t, write $Y^{[t]}$ for the restriction of X to $[0,t]\times\Omega$, which is $\mathcal{B}_t\otimes\mathcal{F}_t$-measurable. Adapt the warmup argument to prove that $Y^{[t]}(\tau(\omega)\wedge t,\omega)$ is $\mathcal{F}_t$-measurable. Show that

$$X(\tau(\omega),\omega)\{\tau(\omega)\le t\} = Y(\tau(\omega)\wedge t,\omega)\{\tau(\omega)\le t\}.$$

Conclude that $X(\tau(\omega),\omega)\{\tau(\omega)<\infty\}$ is $\mathcal{F}_\tau$-measurable.

(iii) Show that an adapted process with right-continuous sample paths is progressively measurable. Argue as follows, for a fixed t. Define $t_{i,n} := it/n$ and, for $0\le s\le t$,

$$X_n(s,\omega) := X(0,\omega)\{s=0\} + \sum\nolimits_{i=1}^{n} X(t_{i,n},\omega)\{t_{i-1,n}<s\le t_{i,n}\}.$$

Show that X_n is $\mathcal{B}_t\otimes\mathcal{F}_t$-measurable and X_n converges pointwise to the restriction of X to $[0,t]\times\Omega$.

$\boxed{\text{Lect 8, Wednesday 3 Feb}}$

4.4 Optional processes

This section is a bit of a detour. I insert it here to get you used to λ-space arguments (see Appendix A).

The ***optional sigma-field*** $\mathcal{O}$ is defined to be the sigma-field on $\mathbb{R}^+\times\Omega$ generated by the set of all cadlag adapted processes. A stochastic process $\{X(t,\omega): t\in\mathbb{R}^+, \omega\in\Omega\}$ is said to be ***optional*** if it is $\mathcal{O}$-measurable.

Remark. I don't yet see the role of the existence of left limits for the generating processes. I am following D&M, trusting there will eventually be some subtle fact that depends on left limits.

Show that each optional process is progressively measurable.

(i) Why is it enough to prove the result for uniformly bounded optional processes? Here bounded means $\sup_{t,\omega}|X(t,\omega)|<\infty$.

(ii) Let $\mathcal{H}$ be the set of all bounded, optional processes that are progressively measurable. Let $\mathcal{G}$ be the set of all bounded, adapted processes with cadlag sample paths.

(iii) Show that $\mathcal{H}$ is a λ-space containing $\mathcal{G}$. Show that $\mathcal{G}$ s stable under pairwise products.

(iv) Invoke a theorem from Appendix A.

4.5 First passage times (a.k.a. debuts)

In discrete time, for each set set $B \in \mathcal{B}(\mathbb{R})$ and process $\{X_n : n \in \mathbb{N}\}$ adapted to a filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$ the random variable

$$\tau_B(\omega) = \inf\{n : X_n(\omega) \in B\}$$

is a stopping time because

$$\{\tau \leq k\} = \cup_{n \leq k}\{X_n \in B\} \in \mathcal{F}_k \qquad \text{for each } k \in \mathbb{N}.$$

As usual, $\inf \emptyset := +\infty$.

In continuous time ($T = \mathbb{R}^+$) the argument becomes much more delicate. Suppose $\{X_t : t \in T\}$ is adapted to $\{\mathcal{F}_t : t \in T\}$ and that $B \in \mathcal{B}(\mathbb{R})$. Define the debut

$$\tau_B(\omega) = \inf\{t \in \mathbb{R}^+ : X(t, \omega) \in B\}.$$

(i) Suppose B is open and X has right-continuous paths. Let S be a countable, dense subset of $\mathbb{R}^+$. Show that

$$\{\omega : \tau_B(\omega) < t\} = \cup_{t > s \in S}\{X_s(\omega) \in B\} \in \mathcal{F}_t$$

Deduce that τ_B is a stopping time if the filtration is right continuous.

(ii) Suppose B is closed and X has continuous paths Define open sets $G_i := \{x : d(x, B) < i^{-1}\}$. Define $\tau_i = \inf\{t : X_t \in G_i\}$. Show that $\tau = \sup_i \tau_i$, so that $\{\tau \leq t\} = \cap_{i \in \mathbb{N}}\{\tau_i \leq t\} \in \mathcal{F}_t$.

In general, the stopping time property depends on a deep measure theory fact:

<1> **Theorem.** *Suppose $(\Omega, \mathcal{G}, \mathbb{P})$ is a complete probability space. Let A be a $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{G}$-measurable subset of $\mathbb{R}^+ \times \Omega$. Then the projection*

$$\pi_\Omega A := \{\omega \in \Omega : (t, \omega) \in A \textit{ for some } t \textit{ in } \mathbb{R}^+\}$$

belongs to $\mathcal{G}$.

If you are interested in the details, see the Appendix on analytic sets, which is based on Dellacherie and Meyer (1978, Chapter III, paras 1–33, 44–45). Recently Bass (2010) wrote a paper that claims to develop the necessary theory without all the technicalities of analytic sets and Choquet capacities. I have not finished reading the paper.

- Suppose B is a Borel set and X is progressively measurable with respect to a standard filtration. Then the debut τ_B is a stopping time. The idea is that the set
$$D_t := \{(s,\omega) : s < t \text{ and } X(s,\omega) \in B\}$$
is $\mathcal{B}_t \otimes \mathcal{F}_t$-measurable. The set $\{\tau < t\}$ equals the projection $\pi_\Omega D_t$. From Theorem 1, this projection belongs to $\mathcal{F}_t$. Thus τ is a stopping time.

4.6 Preservation of martingale properties at stopping times

In discrete time, martingales and stopping times fit together cleanly.

<2> **Theorem.** ***[Stopping Time Lemma: discrete time]*** *Suppose σ and τ are stopping times for a filtration $\{\mathcal{F}_t : t \in T\}$, with T finite. Suppose both stopping times take only values in T. Let F be a set in $\mathcal{F}_\sigma$ for which $\sigma(\omega) \le \tau(\omega)$ when $\omega \in F$. If $\{X_t : t \in T\}$ is a submartingale, then $\mathbb{P}X_\sigma F \le \mathbb{P}X_\tau F$. For supermartingales, the inequality is reversed. For martingales, the inequality becomes an equality.*

Proof See Pollard (2001, page 145).

□

For the analog in continuous time you will need to recall the concept of ***uniform integrability***. See Pollard (2001, Sections 2.8 and 6.6).

<3> **Definition.** *A sequence of integrable random variables $\{Z_n : n \in \mathbb{N}\}$ is said to be uniformly integrable if $\limsup_{n\to\infty} \mathbb{P}|Z_n|\{\,|Z_n| > K\} \to 0$ as $K \to \infty$.*

<4> **Theorem.** *Let $\{Z_n : n \in \mathbb{N}\}$ be a sequence of integrable random variables. The following two conditions are equivalent.*

(i) The sequence is uniformly integrable and it converges in probability to a random variable Z_∞, which is necessarily integrable.

(ii) The sequence converges in $\mathcal{L}^1$ norm, $\mathbb{P}|Z_n - Z_\infty| \to 0$, with a limit Z_∞ that is necessarily integrable.

<5> **Theorem.** ***[Stopping Time Lemma: continuous time]***
Suppose $\{(X_t, \mathcal{F}_t) : 0 \le t \le 1\}$ is a positive supermartingale with cadlag sample paths. Suppose σ and τ are stopping times with $0 \le \sigma(\omega) \le \tau(\omega) \le 1$ for all ω. If $F \in \mathcal{F}_\sigma$ then $\mathbb{P}X_\sigma F \ge \mathbb{P}X_\tau F$.

Proof For each $n \in \mathbb{N}$ define $\sigma_n = 2^{-n}\lceil 2^n \sigma \rceil$. That is,

$$\sigma_n(\omega) = 0\{\sigma(\omega) = 0\} + \sum_{i=1}^{2^n} i/2^n\{(i-1)/2^n < \sigma(\omega) \le i/2^n\}$$

(i) Check that σ_n is a stopping time taking values in a finite subset of $[0,1]$. Question: If we rounded down instead of up, would we still get a stopping time? Check that $F \in \mathcal{F}(\sigma_n)$:

$$F\{\sigma_n \le i/2^n\} = F\{\sigma \le i/2^n\} \in \mathcal{F}(i/2^n).$$

Define τ_n analogously.

(ii) From the discrete case, deduce that

$$\mathbb{P}X(\sigma_n)F \ge \mathbb{P}X(\tau_n)F \qquad \text{for each } n.$$

(iii) Show that $\sigma_n(\omega) \downarrow \sigma(\omega)$ and $\tau_n(\omega) \downarrow \tau(\omega)$ as $n \to \infty$.

(iv) Use right-continuity of the sample path of $X(\cdot, \omega)$ to deduce that $X(\sigma_n, \omega) \to X(\sigma, \omega)$ and $X(\tau_n, \omega) \to X(\tau, \omega)$ for each ω.

left-continuous paths wouldn't help—why not?

(v) Write Z_n for $X(\sigma_n)$. It is enough to show that Z_n converges in L^1 to $Z_\infty := X(\sigma)$, together with a similar assertion about the $X(\tau_n)$ sequence.

(vi) Show that $\{Z_n\}$ is uniformly integrable. Define $\mathcal{G}_i := \mathcal{F}(\sigma_i)$.

(a) First show that $\mathbb{P}Z_n \uparrow c_0 \le \mathbb{P}X_0$ as $n \to \infty$.

(b) Choose m so that $\mathbb{P}Z_m > c_0 - \epsilon$. For a fixed $n \ge m$, show that $\{(Z_i, \mathcal{G}_i) : i = n, n-1, \ldots, m\}$ is a superMG.

(c) For constant K and $n \ge m$, show that

$$\begin{aligned} \mathbb{P}Z_n\{Z_n \ge K\} &= \mathbb{P}Z_n - \mathbb{P}Z_n\{Z_n < K\} \\ &\le c_0 - \mathbb{P}Z_m\{Z_n < K\} \\ &\le \epsilon + \mathbb{P}Z_m\{Z_n \ge K\} \end{aligned}$$

(d) Show that $\mathbb{P}\{Z_n \ge K\} \le c_0/K$, then complete the proof of uniform integrability.

(vii) Prove similarly that $\{X(\tau_n) : n \in \mathbb{N}\}$ is uniformly integrable. (Do we really need the details?) Pass to the limit in the "discretized version" to complete the proof.

□

See Project4_corrected.pdf for a version of this Section that is true (I hope).

4.7 Cadlag versions of (sub-, super-) martingales

Suppose $\{(X_t, \mathcal{F}_t) : 0 \le t \le 1\}$ is a nonnegative supermartingale. For reasons that will soon become apparent, assume that the map $t \mapsto \mathbb{P}X_t$ is right-continuous. Do not assume that the filtration is standard. Instead, write $\{\widetilde{\mathcal{F}}_t : 0 \le t \le 1\}$ for the standard augmented filtration, as defined in Section 1.

See Pollard (2001, Appendix E) for a very condensed account of what follows. I would be happy if you could push the argument through just for martingales.

Not true as stated. See corrected version of Project 4.

The following argument will show that there exists a nonnegative supermartingale $\{(\widetilde{X}_t, \widetilde{\mathcal{F}}_t) : 0 \le t \le 1\}$ with cadlag sample paths such that

$$\mathbb{P}\{\omega : \widetilde{X}_t(\omega) \ne X_t(\omega)\} = 0 \qquad \text{for each fixed } t \text{ in } [0,1].$$

Remarks. The $\widetilde{X}$ process is called a ***version*** of the X process. Note that the set $\Omega_t := \{\omega : \widetilde{X}_t(\omega) \ne X_t(\omega)\}$ is $\mathbb{P}$-negligible, but there is no guarantee that $\cup_{0 \le t \le 1}\Omega_t$ is $\mathbb{P}$-negligible.

Note also that the desire to have cadlag paths forces us to work with the larger filtration.

(i) Start from a "dense skeleton" $\{X_s : s \in S\}$ where S is a countable dense subset of $[0,1]$. Suppose $S := \cup_{k \in \mathbb{N}} S_k$ for an increasing sequence $\{S_k\}$ of finite subsets of $[0,1]$. It might help to insist that $1 \in S_1$.

(ii) Use Lemma 2 to show that

$$\mathbb{P}\{\max_{s \in S_k} X_s > x\} \le \mathbb{P}X_0/x \qquad \text{for each } x > 0.$$

Let k tend to infinity then x tend to infinity to deduce that the set $\Omega_\infty := \{\omega : \sup_{s \in S} X_s(\omega) < \infty\}$ has probability one.

(iii) For fixed rational numbers $0 < \alpha < \beta$, invoke Dubin's inequality (Pollard 2001, Theorem 6.20) to show that the event

$$\begin{aligned} &A(\alpha, \beta, k, n) \\ &\quad := \{\text{the process } \{X_s : s \in S_k\} \text{ makes at least } n \text{ upcrossings of } [a, \beta]\ \} \end{aligned}$$

has probability less than $(\alpha/\beta)^n$.

(iv) Let k tend to infinity, then n tend to infinity, then take a union over rational pairs to deduce existence of an $N \in \mathcal{N}$ such that, for $\omega \in N^c$, the sample path $X(\cdot, \omega)$ (as a function on S) is bounded and

$$X(\cdot, \omega) \text{ makes only finitely many upcrossings of each rational interval}.$$

(v) Deduce that $\widetilde{X}_t(\omega) := \lim_{s\downarrow\downarrow t} X(s,\omega)$ exists and is finite for each $t \in [0,1)$ and each $\omega \in N^c$. Deduce also that $\lim_{s\uparrow\uparrow t} X(s,\omega)$ exists and is finite for each $t \in (0,1]$ and each $\omega \in N^c$.

$\uparrow\uparrow$ means strictly increasing and $\downarrow\downarrow$ means strictly decreasing

(vi) Define $\widetilde{X}(\cdot,\omega) \equiv 0$ for $\omega \in N$. Show that $\widetilde{X}$ has cadlag sample paths.

(vii) Note: $\widetilde{X}_t$ need not be $\mathcal{F}_t$-measurable but it is measurable with respect to the sigma-field $\widetilde{\mathcal{F}}_t$.

False in generality implied

(viii) Show that $\widetilde{X}_t = X_t$ almost surely, for each fixed t. Hint: It might help to think about what would happen if you repeated the construction with S replaced by $S \cup \{t\}$, for a fixed t.

(ix) Show that $\{(\widetilde{X}_t, \widetilde{\mathcal{F}}_t) : 0 \le t \le 1\}$ is a supermartingale with cadlag sample paths. This where the right-continuity of $t \mapsto \mathbb{P}X_t$ is needed. You might wnt to look at Pollard (2001, page 335) for hints.

Not necessarily. See corrected version a slightly weaker true assertion

(x) Is it true that $\widetilde{X}$ is a version of X?

How could we extend the argument to get versions of submartingales $\{X_t : t \in \mathbb{R}^+\}$ with cadlag sample paths?

To complete your understanding, you might try to find a filtration (which is necessarily not standard) for which there is a martingale that does not have a version with cadlag sample paths.

Why do you think that most authors prefer to assume the usual conditions?

References

Bass, R. F. (2010, January). The measurability of hitting times. arXiv:1001.3619v1.

Dellacherie, C. and P. A. Meyer (1978). *Probabilities and Potential*. Amsterdam: North-Holland. (First of three volumes).

Dudley, R. M. (1989). *Real Analysis and Probability*. Belmont, Calif: Wadsworth.

Pollard, D. (2001). *A User's Guide to Measure Theoretic Probability*. Cambridge University Press.

Appendix A

Lambda spaces and generating classes

You will be using many λ-space arguments. What follows is extracted from a rewrite of Pollard (2001, Section 2.11). Look in the Handouts subdirectory of the website site for

http://www.stat.yale.edu/∼pollard/Courses/600.spring2010/

for a version with proofs.

<1> **Definition.** *Let $\mathcal{H}$ be a set of bounded, real-valued functions on a set $\mathcal{X}$. Call $\mathcal{H}$ a **λ-space** if:*

(i) $\mathcal{H}$ is a vector space

(ii) each constant function belongs to $\mathcal{H}$;

(iii) if $\{h_n\}$ is an increasing sequence of functions in $\mathcal{H}$ whose pointwise limit h is bounded then $h \in \mathcal{H}$.

Remember that $\sigma(\mathcal{H})$ is the smallest σ-field on $\mathcal{X}$ for which each h in $\mathcal{H}$ is $\sigma(\mathcal{H})\backslash\mathcal{B}(\mathbb{R})$-measurable. It is the σ-field generated by the collection of all sets of the form $\{h \in B\}$ with $h \in \mathcal{H}$ and $B \in \mathcal{B}(\mathbb{R})$.

<2> **Lemma.** *If a λ-space $\mathcal{H}$ is stable under the formation of pointwise products of pairs of functions then it consists of all bounded, $\sigma(\mathcal{H})$-measurable functions.*

<3> **Theorem.** *Let $\mathcal{G}$ be a set of functions from a λ-space $\mathcal{H}$. If $\mathcal{G}$ is stable under the formation of pointwise products of pairs of functions then $\mathcal{H}$ contains all bounded, $\sigma(\mathcal{G})$-measurable functions.*

Appendix B

Analytic sets

B.1 Overview

For a discrete-time process $\{X_n\}$ adapted to a filtration $\{\mathcal{F}_n : n \in \mathbb{N}\}$, the prime example of a stopping time is $\tau = \inf\{n \in \mathbb{N} : X_n \in B\}$, the first time the process enters some Borel set B. For a continuous-time process $\{X_t\}$ adapted to a filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$, it is less obvious whether the analogously defined random variable $\tau = \inf\{t : X_t \in B\}$ is a stopping time. (Also it is not necessarily true that X_τ is a point of B.) The most satisfactory resolution of the underlying measure-theoretic problem requires some theory about analytic sets. What follows is adapted from Dellacherie and Meyer (1978, Chapter III, paras 1–33, 44–45). The following key result will be proved in this handout.

<1> **Theorem.** *Let A be a $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$-measurable subset of $\mathbb{R}^+ \times \Omega$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Then:*

(i) The projection $\pi_\Omega A := \{\omega \in \Omega : (t, \omega) \in A$ for some t in $\mathbb{R}^+\}$ belongs to $\mathcal{F}$.

(ii) There exists an $\mathcal{F}$-measurable random variable $\psi : \Omega \to \mathbb{R}^+ \cup \{\infty\}$ such that $\psi(\omega) < \infty$ and $(\psi(\omega), \omega) \in A$ for almost all ω in the projection $\pi_\Omega A$, and $\psi(\omega) = \infty$ for $\omega \notin \pi_\Omega A$.

Remark. The map ψ in (ii) is called a ***measurable cross-section*** of the set A. Note that the cross-section $A_\omega := \{t \in \mathbb{R}^+ : (t, \omega) \in A\}$ is empty when $\omega \notin \pi_\Omega A$. It would be impossible to have $(\psi(\omega), \omega) \in A$ for such an ω.

The proofs will exploit the properties of the collection of analytic subsets of $[0, \infty] \times \Omega$. As you will see, the analytic sets have properties analogous to those of sigma-fields—stability under the formation of countable unions and intersections. They are not necessarily stable under complements, but they do have an extra stability property for projections that is not shared by measurable sets. The Theorem is made possible by the fact that the product-measurable subsets of $\mathbb{R}^+ \times \Omega$ are all analytic.

B.2 Notation

A collection $\mathcal{D}$ of subsets of a set $\mathcal{X}$ with $\emptyset \in \mathcal{D}$ is called a ***paving*** on $\mathcal{X}$. A paving that is closed under the formation of unions of countable subcollections is said to be a $\cup c$-paving. For example, the set $\mathcal{D}_\sigma$ of all unions of countable subcollections of $\mathcal{D}$ is a $\cup c$-paving. Similarly, the set $\mathcal{D}_\delta$ of all intersections of countable subcollections of $\mathcal{D}$ is a $\cap c$-paving. Note that $\mathcal{D}_{\sigma\delta} := (\mathcal{D}_\sigma)_\delta$ is a $\cap c$-paving but it need not be stable under $\cup c$.

Let T be a compact metric space equipped with the paving $\mathcal{K}(T)$ of compact subsets and its Borel sigma-field $\mathcal{B}(T)$, which is generated by $\mathcal{K}(T)$.

Remark. In fact, $\mathcal{K}(T)$ is also the class of closed subsets of the compact T.

For Theorem <1>, the appropriate space will be $T = [0, \infty]$. The sets in $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$ can be identified with sets in $\mathcal{B}(T) \otimes \mathcal{F}$. The compactness of T will be needed to derive good properties for the projection map $\pi_\Omega : T \times \Omega \to \Omega$.

An important role will be played by the $\cap f$-paving

$$\mathcal{K}(T) \times \mathcal{F} := \{K \times F : K \in \mathcal{K}(T),\, F \in \mathcal{F}\} \qquad \text{on } T \times \Omega$$

and by the paving $\mathcal{R}$ that consists of all finite unions of sets from $\mathcal{K}(T) \times \mathcal{F}$. That is, $\mathcal{R}$ is the $\cup f$-closure of $\mathcal{K}(T) \times \mathcal{F}$. Note (Problem [1]) that $\mathcal{R}$ is a $(\cup f, \cap f)$-paving on $T \times \Omega$. Also, if $R = \cup_i K_i \times F_i$ then, assuming we have discarded any terms for which $K_i = \emptyset$,

$$\pi_\Omega(R) = \cup_i \pi_\Omega \big(K_i \times F_i\big) = \cup_i F_i \in \mathcal{F}.$$

Remark. If $\mathcal{E}$ and $\mathcal{F}$ are sigma-fields, note the distinction between

$$\mathcal{E} \times \mathcal{F} = \{E \times F : E \in \mathcal{E},\, F \in \mathcal{F}\}$$

and $\mathcal{E} \otimes \mathcal{F} := \sigma(\mathcal{E} \times \mathcal{F})$.

B.3 Why compact sets are needed

Many of the measurability difficulties regarding projections stem from the fact that they do not "preserve set-theoretic operations" in the way that inverse images do: $\pi_\Omega \big(\cup_i A_i\big) = \cup_i \pi_\Omega A_i$ but $\pi_\Omega \big(\cap_i A_i\big) \subseteq \cap_i \pi_\Omega A_i$. Compactness of cross-sections will allow us to strengthen the last inclusion to an equality.

<2> **Lemma.** *[Finite intersection property] Suppose $\mathcal{K}_0$ is a collection of compact subsets of a metric space $\mathcal{X}$ for which each finite subcollection has a nonempty intersection. Then $\cap \mathcal{K}_0 \neq \emptyset$.* also works for any Hausdorff topological space

Proof Arbitrarily choose a K_0 from $\mathcal{K}_0$. If $\cap\mathcal{K}_0$ were empty then the sets $\{K^c : K \in \mathcal{K}_0\}$ would be an open cover of K_0. Extract a finite subcover $\cup_{i=1}^m K_i^c$. Then $\cap_{i=0}^m K_i = \emptyset$, a contradiction.

□

<3> **Corollary.** *Suppose $\{A_i : i \in \mathbb{N}\}$ is a decreasing sequence of subsets of $T\times\Omega$ for which each ω-cross-section $K_i(\omega) := \{t \in T : (t,\omega) \in A_i\}$ is compact. Then $\pi_\Omega\big(\cap_{i\in\mathbb{N}} A_i\big) = \cap_{i\in\mathbb{N}}\pi_\Omega A_i$.*

Proof Suppose $\omega \in \cap_{i\in\mathbb{N}}\pi_\Omega A_i$. Then $\{K_i(\omega) : i \in \mathbb{N}\}$ is a decreasing sequence of compact, nonempty (because $\omega \in \pi_\Omega A_i$) subsets of T. The finite intersection property of compact sets ensures that there is a t in $\cap_{i\in\mathbb{N}}K_i(\omega)$. The point (t,ω) belongs to $\cap_{i\in\mathbb{N}}A_i$ and $\omega \in \pi_\Omega\big(\cap_i A_i\big)$.

□

Remark. For our applications, we will be dealing only with sequences, but the argument also works for more general collections of sets with compact cross-sections.

<4> **Corollary.** *If $B = \cap_{i\in\mathbb{N}}R_i$ with $R_i \in \mathcal{R}$ then $\pi_\Omega B = \cap_{i\in\mathbb{N}}\pi_\Omega R_i \in \mathcal{F}$.*

Proof Note that the cross-section of each $\mathcal{R}$-set is a finite union of compact sets, which is compact. Without loss of generality, we may assume that $R_1 \supseteq R_2 \supseteq \dots$. Invoke Corollary <3>.

□

B.4 Measurability of some projections

For which $B \in \mathcal{B}(T)\otimes\mathcal{F}$ is it true that $\pi_\Omega(B) \in \mathcal{F}$? From Corollary <4>, we know that it is true if B belongs to $\mathcal{R}_\delta$. The following properties of outer measures (see Problem [2]) will allow us to extend this nice behavior to sets in $\mathcal{R}_{\sigma\delta}$:

(i) If $A_1 \subseteq A_2$ then $\mathbb{P}^*(A_1) \le \mathbb{P}^*(A_2)$

(ii) If $\{A_i : i \in \mathbb{N}\}$ is an increasing sequence then $\mathbb{P}^*\big(A_i\big) \uparrow \mathbb{P}^*\big(\cup_{i\in\mathbb{N}} A_i\big)$.

(iii) If $\{F_i : i \in \mathbb{N}\} \subseteq \mathcal{F}$ is a decreasing sequence then

$$\mathbb{P}^*\big(F_i\big) = \mathbb{P}F_i \downarrow \mathbb{P}\big(\cap_{i\in\mathbb{N}} F_i\big) = \mathbb{P}^*\big(\cap_{i\in\mathbb{N}} F_i\big).$$

For each subset D of $T\times\Omega$ define $\Psi^*(D) := \mathbb{P}^*\pi_\Omega D$, the outer measure of the projection of D onto Ω. If $D_i \uparrow D$ then $\pi_\Omega D_i \uparrow \pi_\Omega D$. If $R_i \in \mathcal{R}$ and $R_i \downarrow B$ then $\pi_\Omega R_i \in \mathcal{F}$ and $\pi_\Omega R_i \downarrow \pi_\Omega B \in \mathcal{F}$. The properties for $\mathbb{P}^*$ carry over to analogous properties for Ψ^*:

(i) If $D_1 \subseteq D_2$ then $\Psi^*(D_1) \le \Psi^*(D_2)$

(ii) If $\{D_i : i \in \mathbb{N}\}$ is an increasing sequence then $\Psi^*\big(D_i\big) \uparrow \Psi^*\big(\cup_{i\in\mathbb{N}} D_i\big)$.

(iii) If $\{R_i : i \in \mathbb{N}\} \subseteq \mathcal{R}$ is a decreasing sequence then $\Psi^*\big(R_i\big) \downarrow \Psi^*\big(\cap_{i\in\mathbb{N}} R_i\big)$.

With just these properties, we can show that π_Ω behaves well on a much larger collection of sets than $\mathcal{R}$.

<5> **Lemma.** *If $A \in \mathcal{R}_{\sigma\delta}$ then $\Psi^*(B) = \sup\{\Psi^*(B) : B \in \mathcal{R}_\delta\}$. Consequently, the set $\pi_\Omega A$ belongs to $\mathcal{F}$.*

Proof Write A as $\cap_{i\in\mathbb{N}} D_i$ with $D_i = \cup_{j\in\mathbb{N}} R_{ij} \in \mathcal{R}_\sigma$. As $\mathcal{R}$ is $\cup f$-stable, we may assume that R_{ij} is increasing in j for each fixed i.

Suppose $\Psi^*(A) > M$ for some constant M. Invoke (ii) for the sequence $\{AR_{1j}\}$, which increases to $AD_1 = A$, to find an index j_1 for which the set $R_1 := R_{1j_1}$ has $\Psi^*\big(AR_1\big) > M$.

The sequence $\{AR_1R_{2j}\}$ increases to $AR_1D_2 = AR_1$. Again by (ii), there exists an index j_2 for which the set $R_2 = R_{2j_2}$ has $\Psi^*(AR_1R_2) > M$. And so on. In this way we construct sets R_i in $\mathcal{R}$ for which

$$\Psi^*(R_1R_2\dots R_n) \ge \Psi^*(AR_1R_2\dots R_n) > M$$

for every n. The set $B_M := \cap_{i\in\mathbb{N}} R_i$ belongs to $\mathcal{R}_\delta$; it is a subset of $\cap_{i\in\mathbb{N}} D_i = A$; and, by (iii), $\Psi^*(B) \ge M$.

By Corollary <4>, the set B_M projects to a set $F_M := \pi_\Omega B_M$ in $\mathcal{F}$ and hence $\mathbb{P}F_M = \Psi^* B \ge M$. The set $\pi_\Omega A$ is inner regular, in the sense that

$$\mathbb{P}^*\pi_\Omega A = \Psi^* A = \sup\{\mathbb{P}F : \pi_\Omega A \supseteq F \in \mathcal{F}\}$$

It follows (Problem [2]) that the set $\pi_\Omega A$ belongs to $\mathcal{F}$.

□

The properties shared by $\mathbb{P}^*$ and Ψ^* are so useful that they are given a name.

<6> **Definition.** *Suppose $\mathcal{S}$ is a paving on a set S. A function Ψ defined for all subsets of S and taking values in $[-\infty, \infty]$ is said to be a* ***Choquet $\mathcal{S}$-capacity*** *if it satisfies the following three properties.*

(i) (i) If $D_1 \subseteq D_2$ then $\Psi(D_1) \le \Psi(D_2)$

(ii) (ii) If $\{D_i : i \in \mathbb{N}\}$ is an increasing sequence then $\Psi\big(D_i\big) \uparrow \Psi\big(\cup_{i\in\mathbb{N}} D_i\big)$.

(iii) (iii) If $\{S_i : i \in \mathbb{N}\} \subseteq \mathcal{S}$ *is a decreasing sequence then* $\Psi\big(S_i\big) \downarrow \Psi\big(\cap_{i\in\mathbb{N}} S_i\big)$.

The outer measure $\mathbb{P}^*$ is a Choquet $\mathcal{F}$-capacity defined for the subsets of Ω. Moreover, if Ψ is any Choquet $\mathcal{F}$-capacity defined for the subsets of Ω then $\Psi^*(D) := \Psi(\pi_\Omega D)$ is a Choquet $\mathcal{R}$-capacity defined for the subsets of $T \times \Omega$. The argument from Lemma <5> essentially shows that if $A \in \mathcal{R}_{\sigma\delta}$ then $\Psi^*(B) = \sup\{\Psi^*(B) : B \in \mathcal{R}_\delta\}$ for every such Ψ^*, whether defined via $\mathbb{P}^*$ or not.

B.5 Analytic sets

The paving of $\mathcal{S}$-analytic sets can be defined for any paving $\mathcal{S}$ on a set S. For our purposes, the most important case will be $S = T \times \Omega$ with $\mathcal{S} = \mathcal{R}$.

<7> **Definition.** *Suppose* $\mathcal{S}$ *is a paving on a set* S*. A subset* A *of* S *is said to be* $\mathcal{S}$*-analytic if there exists a compact metric space* E *and a subset* D *in* $\big(\mathcal{K}(E) \times \mathcal{S}\big)_{\sigma\delta}$ *for which* $A = \pi_S D$*. Write* $\mathcal{A}(\mathcal{S})$ *for the set of all* $\mathcal{S}$*-analytic subsets of* S*.*

Remark. Note that $\mathcal{R}_{\sigma\delta} = (\mathcal{K}(T) \times \mathcal{F})_{\sigma\delta}$. The σ takes care of the $\cup f$ operation needed to generate $\mathcal{R}$ from $\mathcal{K}(T) \times \mathcal{F}$. The $\mathcal{R}$-analytic sets are also called $\mathcal{K}(T) \times \mathcal{F}$-analytic sets.

In fact, it is possible to find a single E that defines all the $\mathcal{S}$-analytic subsets, but that possibility is not important for my purposes. What is important is the fact that $\mathcal{A}(\mathcal{S})$ is a $(\cup c, \cap c)$-paving: see Problem [3].

When E is another compact metric space, Tychonoff's theorem (see Dudley 1989, Section 2.2, for example) ensures not only that the product space $E \times T$ is a compact metric space but also that $\mathcal{K}(E) \times \mathcal{K}(T) \subseteq \mathcal{K}(E \times T)$.

Lemma <5>, applied to $\widetilde{T} := E \times T$ instead of T and with $\widetilde{\mathcal{R}}$ the $\cup f$-closure of $\mathcal{K}(E \times T) \times \mathcal{F}$, implies that

$$\widetilde{\pi}_\Omega D \in \mathcal{F} \qquad \text{for each } D \text{ in } \widetilde{\mathcal{R}}_{\sigma\delta}.$$

Here $\widetilde{\pi}_\Omega$ projects $E \times T \times \Omega$ onto Ω. We also have

$$\widetilde{\mathcal{R}}_{\sigma\delta} \supseteq \big(\mathcal{K}(E) \times \mathcal{K}(T) \times \mathcal{F}\big)_{\sigma\delta} = \big(\mathcal{K}(E) \times \mathcal{R}\big)_{\sigma\delta}$$

where $\mathcal{R}$ is the $\cup f$-closure of $\mathcal{K}(T) \times \mathcal{F}$, as in Section **??**. As a special case of property <5> we have

$$\widetilde{\pi}_\Omega D \in \mathcal{F} \qquad \text{for each } D \text{ in } \big(\mathcal{K}(E) \times \mathcal{R}\big)_{\sigma\delta}.$$

Write $\widetilde{\pi}_\Omega$ as a composition of projection $\pi_\Omega \circ \widetilde{\pi}_{T\times\Omega}$, where $\widetilde{\pi}_{T\times\Omega}$ projects $E \times T \times \Omega$ onto $T \times \Omega$. As E ranges over all compact metric spaces and D ranges over all the $\big(\mathcal{K}(E) \times \mathcal{R}\big)_{\sigma\delta}$ sets, the projections $A := \widetilde{\pi}_{T\times\Omega} D$ range over all $\mathcal{R}$-analytic subsets of $T \times \Omega$. Property <5> is equivalent to the assertion

$$\pi_\Omega A \in \mathcal{F} \qquad \text{for all } A \in \mathcal{A}(\mathcal{R}).$$

In fact, the method used to prove Lemma <5> together with an analogue of the argument just outlined establishes an approximation theorem for analytic sets and general Choquet capacities.

<8> **Theorem.** *Suppose $\mathcal{S}$ is a $(\cup f, \cap f)$-paving on a set S and Let Ψ is a Choquet $\mathcal{S}$-capacity on S. Then $\Psi(A) = \sup\{\Psi(B) : A \supseteq B \in \mathcal{S}_\delta\}$. for each A in $\mathcal{A}(\mathcal{S})$.*

To prove assertion (i) of Theorem <1>, we have only to check that

$$\mathcal{B}(T) \otimes \mathcal{F} \subseteq \mathcal{A}(\mathcal{R})$$

for the special case where $T = [0,\infty]$. By Problem [3], $\mathcal{A}(\mathcal{R})$ is a $(\cup c, \cap c)$-paving. It follows easily that

$$\mathcal{H} := \{H \in \mathcal{B}(T) \otimes \mathcal{F} : H \in \mathcal{A}(\mathcal{R}) \text{ and } H^c \in \mathcal{A}(\mathcal{R})\,\}$$

is a sigma-field on $T \times \Omega$. Each $K \times F$ with $K \in \mathcal{K}(T)$ and $F \in \mathcal{F}$ belongs to $\mathcal{H}$ because $\mathcal{K}(T) \times \mathcal{F} \subseteq \mathcal{R} \subseteq \mathcal{A}(\mathcal{R})$ and

$$\begin{aligned}(K \times F)^c &= \big(K \times F^c\big) + \big(K^c \times \Omega\big)\\ K^c &= \cup_{i\in\mathbb{N}}\{t : d(t,K) \geq 1/i\} \in \mathcal{K}(T)_\sigma\end{aligned}$$

It follows that $\mathcal{H} = \sigma(\mathcal{K}(T) \times \mathcal{F}) = \mathcal{B}(T) \otimes \mathcal{F}$ and $\mathcal{B}(T) \otimes \mathcal{F} \subseteq \mathcal{A}(\mathcal{R})$.

B.6 Existence of measurable cross-sections

The general Theorem <8> is exactly what we need to prove part (ii) of Theorem <1>.

Proof Once again identify A with an $\mathcal{R}$-analytic subset of $T \times \Omega$, where $T = [0,\infty]$. The result is trivial if $\alpha_1 := \mathbb{P}\pi_\Omega A = 0$, so assume $\alpha_1 > 0$.

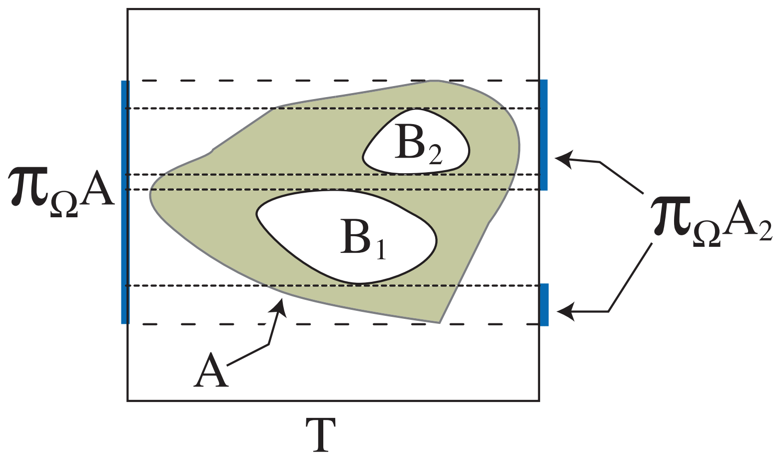
Invoke Theorem <8> for the $\mathcal{R}$-capacity defined by $\Psi^*(D) = \mathbb{P}^*(\pi_\Omega D)$. Find a subset with $A \supseteq B_1 \in \mathcal{R}_\delta$ and $\mathbb{P}\left(\pi_\Omega B_1\right) = \Psi^*(B_1) \geq \alpha_1/2$. Define

$$\psi_1(\omega) := \inf\{t \in \mathbb{R}^+ : (t,\omega) \in B_1\}.$$

Because the set B_1 has compact cross-sections, the infimum is actually achieved for each ω in $\pi_\Omega B_1$. For $\omega \notin \pi_\Omega B_1$ the infimum equals ∞. Define

$$A_2 := \{(t,\omega) \in A : \omega \notin \pi_\Omega B_1\} = A \cap \big(T \times (\pi_\Omega B_1)^c\big)$$

Note that $A_2 \in \mathcal{A}(\mathcal{R})$ and $\alpha_2 := \mathbb{P}\pi_\Omega A_2 \le \alpha_1/2$. Without loss of generality suppose $\alpha_2 > 0$. Find a subset with $A_2 \supseteq B_2 \in \mathcal{R}_\delta$ and $\mathbb{P}\big(\pi_\Omega B_2\big) = \Psi^*(B_2) \ge \alpha_2/2$. Define $\psi_2(\omega)$ as the first hitting time on B_2.



And so on. The sets $\{\pi_\Omega B_i : i \in \mathbb{N}\}$ are disjoint, with $F := \cup_{i\in\mathbb{N}}\pi_\Omega B_i$ a subset of $\pi_\Omega A$. By construction $\alpha_i \downarrow 0$, which ensures that $\mathbb{P}\big((\pi_\Omega A)\backslash F\big) = 0$.

If $\alpha_i = 0$ for some i, the construction requires only finitely many steps.

Define $\psi := \inf_{i\in\mathbb{N}} \psi_i$. On B we have $(\psi(\omega),\omega) \in A$.

□

B.7 Problems

[1] Suppose $\mathcal{S}$ is a paving (on a set S), which is $\cap f$-stable. Let $\mathcal{S}_{\cup f}$ consists of the set of all unions of finite collections of sets from $\mathcal{S}$. Show that $\mathcal{S}_{\cup f}$ is a $(\cup f, \cap f)$-paving. Hint: Show that $(\cup_i S_i) \cap (\cup_j T_j) = \cup_{i,j}(S_i \cap T_j)$.

[2] The outer measure of a set $A \subseteq \Omega$ is defined as $\mathbb{P}A := \inf\{\mathbb{P}F : A \subseteq F \in \mathcal{F}\}$.

(i) Show that the infimum is achieved, that is, there exists an $F \in \mathcal{F}$ for which $A \subseteq F$ and $\mathbb{P}^*A = \mathbb{P}F$. Hint: Consider the intersection of a sequence of sets for which $\mathbb{P}F_n \downarrow \mathbb{P}^*A$.

(ii) Suppose $\{D_n : n \in \mathbb{N}\}$ is an increasing sequence of sets (not necessarily $\mathcal{F}$-measurable) with union D. Show that $\mathbb{P}^*D_n \uparrow \mathbb{P}^*D$. Hint: Find sets with $D_i \subseteq F_i \in \mathcal{F}$ and $\mathbb{P}^* D_i = \mathbb{P} F_i$. Show that $\cap_{i \geq n} F_i \uparrow F \supseteq D$ and $\mathbb{P} F \leq \sup_{i \in \mathbb{N}} \mathbb{P}^* D_i$.

(iii) Suppose D is a subset of Ω for which $\mathbb{P}^* D = \sup\{\mathbb{P} F_0 : D \supseteq F_0 \in \mathcal{F}\}$. Show that D belongs to the $\mathbb{P}$-completion of $\mathcal{F}$ (or to $\mathcal{F}$ itself if $\mathcal{F}$ is $\mathbb{P}$-complete). Hint: Find sets F and F_i in $\mathcal{F}$ for which $F_i \subseteq D \subseteq F$ and $\mathbb{P} F_i \uparrow \mathbb{P}^* D = \mathbb{P} F$. Show that $F \backslash \big(\cup_{i \in \mathbb{N}} F_i \big)$ has zero $\mathbb{P}$-measure.

[3] Suppose $\{A_\alpha : \alpha \in \mathbb{N}\} \subseteq \mathcal{A}(\mathcal{S})$. Show that $\cup_\alpha A_a \in \mathcal{A}(\mathcal{S})$ and $\cap_\alpha A_a \in \mathcal{A}(\mathcal{S})$, by the following steps. Recall that there exist compact metric spaces $\{E_\alpha : \alpha \in \mathbb{N}\}$, each equipped with its paving $\mathcal{K}_\alpha$ of compact subsets, and sets $D_\alpha \in \big(\mathcal{K}_\alpha \times \mathcal{S}\big)_{\sigma\delta}$ for which $A_\alpha = \pi_S D_\alpha$. D&M Theorem 3.8

(i) Define $E := \times_{\alpha \in \mathbb{N}} E_\alpha$ and $E_{-\beta} = \times_{\alpha \in \mathbb{N} \backslash \{\beta\}} E_\alpha$. Show that E is a compact metric space.

(ii) Define $\widetilde{D} := D_\alpha \times E_{-\alpha}$. Show that $\widetilde{D}_\alpha \in \big(\mathcal{K}(E) \times \mathcal{S}\big)_{\sigma\delta}$ and that $A_\alpha = \widetilde{\pi}_S \widetilde{D}_\alpha$, where $\widetilde{\pi}_S$ denotes the projection map from $E \times S$ to S.

(iii) Show that $\cap_\alpha A_\alpha = \widetilde{\pi}_S \Big(\cap_\alpha \widetilde{D}_\alpha \Big)$ and $\cap_\alpha \widetilde{D}_\alpha \in \big(\mathcal{K}(E) \times \mathcal{S}\big)_{\sigma\delta}$.

(iv) Without loss of generality suppose the E_α spaces are disjoint—otherwise replace E_α by $\{\alpha\} \times E_\alpha$. Define $H = \cup_{\alpha \in \mathbb{N}} E_\alpha$ and $E^* := H \cup \{\infty\}$. Without loss of generality suppose the metric d_α on E_α is bounded by $2^{-\alpha}$. Define

$$d(x,y) = d(y,x) := \begin{cases} d_\alpha(x,y) & \text{if } x, y \in E_\alpha \\ 2^{-\alpha} + 2^{-\beta} & \text{if } x \in E_\alpha,\ y \in E_\beta \text{ with } \alpha \neq \beta \\ 2^{-\alpha} & \text{if } y = \infty \text{ and } x \in E_\alpha \end{cases}$$

Show that E^* is a compact metric space under d.

(v) Suppose $D_\alpha = \cap_{i \in \mathbb{N}} B_{\alpha i}$ with $B_{\alpha i} \in (\mathcal{K}_\alpha \times \mathcal{S})_\sigma$. Show that $\cup_\alpha D_\alpha = \cap_i \cup_\alpha B_{\alpha,i}$. Hint: Consider the intersection with $E_\alpha \times S$.

(vi) Deduce that $\cup_\alpha D_\alpha \in (\mathcal{K}(E^*) \times \mathcal{S})_{\sigma\delta}$.

(vii) Conclude that $\cup_\alpha A_\alpha = \pi_S \cup_\alpha D_\alpha \in \mathcal{A}(\mathcal{S})$.