Project 5

Lévy's martingale characterization of Brownian Motion

Lect 10, Wednesday 10 Feb

I believe the following theorem explains why Brownian motion plays such a central role in stochastic calculus. I will say more about this belief when we come to diffusions.

<1> **Theorem.** *(Lévy 1948, pages 77-78 of second edition)*
Suppose $\{(M_t, \mathcal{F}_t) : 0 \le t \le 1\}$ is a martingale with continuous sample paths and $M_0 = 0$. Suppose also that $M_t^2 - t$ is a martingale. Then M is a Brownian motion.

5.1 Proof of Lévy's theorem

We need to show that the finite-dimensional distributions of M agree with those for a Brownian motion. The main ideas are contained in the proof that $M_1 \sim N(0,1)$.

You will be needing some bounds on remainder terms in series expansions.

<2> **Lemma.** *For each real x define*

$$R_1(x) := e^x(1-x) - 1 \qquad \textit{and} \qquad R_2(x) := e^{ix} - 1 - (ix) - \tfrac{1}{2}(ix)^2.$$

Then there exists a finite constant C such that $|R_1(x)| \le Cx^2 e^{|x|}$ and $|R_2(x) \le C|x|^3$ for all x.

> **Remark.** I believe $C = 1/2$ would suffice for R_1 and $C = 1/3$ would suffice for R_2.

How to show $M_1 \sim N(0,1)$.

(i) Cut each sample path of M into small increments.

Take $\tau_{n,0} = 0$ and

$$\tau_{n,j+1} = 1 \wedge \left(n^{-1} + \tau_{n,j}\right) \wedge \inf\{t \ge \tau_{n,j} : |M(t) - M(\tau_{n,j})| \ge n^{-1}\}$$

For $j = 1, 2, \dots$ define random variables $\xi_{n,j} := M(\tau_{n,j}) - M(\tau_{n,j-1})$ and $\delta_{n,j} := \tau_{n,j} - \tau_{n,j-1}$ and $v_{n,j} := \mathbb{P}(\xi_{n,j}^2 \mid \mathcal{F}(\tau_{n,j}))$.

(ii) Why is each $\tau_{n,j}$ a stopping time? See Problem [1] if you really want to understand some of the details.

(iii) Show that $\max_j |\xi_{n,j}| \le n^{-1}$ and $\max_j \delta_{n,j} \le n^{-1}$.

(iv) Show that there exist an increasing sequence of integers $\{k(n)\}$ such that $\mathbb{P}\{\tau_{n,k(n)} \neq 1\} \to 0$ as $n \to \infty$. Hint: Use the uniform continuity of the sample path $M(\cdot,\omega)$ to show that $\tau_{n,k}(\omega) = 1$ for some finite $k = k(\omega)$.

> The notation is getting too cluttered. To simplify, omit most of the subscript n's and other messy symbols when carrying out calculations for a fixed n. For example, write M_j for $M(\tau_{n,j})$, and $\mathcal{F}_j$ for $\mathcal{F}(\tau_{n,j})$, and $\mathbb{P}_j$ for $\mathbb{P}(\cdots \mid \mathcal{F}(\tau_{n,j}))$.

(v) For fixed n, use the Stopping Time Lemma (for continuous time) to show that $\mathbb{P}_{j-1}\xi_j = 0$ almost surely.

(vi) For fixed n, show that

$$\mathbb{P}_{j-1}\left((M_{j-1}+\xi_j)^2 - \tau_j\right) = M_{j-1}^2 - \tau_{j-1}.$$

Deduce that $v_j = \mathbb{P}_{j-1}\delta_j$.

(vii) Show that

$$\mathbb{P}\left|\sum\nolimits_{j\le k(n)}(\delta_j - v_j)\right|^2 = \sum\nolimits_{j\le k(n)}\mathbb{P}(\delta_j - v_j)^2 \le \sum\nolimits_{j\le k(n)}\mathbb{P}\delta_j^2 \le n^{-1}.$$

Deduce that $\sum_{j\le k(n)} v_j \to 1$ in probability as $n\to\infty$.

(viii) Define $\sigma_n := \max\{j : \sum_{\ell\le j} v_\ell \le 2\}$. Show that σ_n is a stopping time for the $\{\mathcal{F}(\tau_{n,j}) : j = 0,1,\dots\}$ filtration.

Remark. Here you will need to use these facts: the partial sums of the v_ℓ's form an increasing sequence; and v_ℓ is $\mathcal{F}_{\ell-1}$-measurable. In a terminolgy that will soon mean more to you, the partial sums form a ***predictable*** sequence.

> I adapted the rest of the argument from Pollard (1984, Section VIII.1). See the Notes at the end of Chapter VIII of that book and Pollard (2001, Notes to Chapter 9) for more about the origins of the method.

(ix) Define $\eta_j := \eta_{n,j} := \xi_{n,j}\{\sigma_n \ge j\}$ and $w_j := w_{n,j} := v_{n,j}\{\sigma_n \ge j\}$. Show that $\mathbb{P}_{j-1}\eta_j = 0$ and $\mathbb{P}_{j-1}\eta_j^2 = w_j$ and

$$2 \ge \sum\nolimits_{j\le k(n)} w_j \to 1 \qquad \text{in probability.}$$

(x) Explain why it is enough to prove that $\sum_{j\le k(n)} \eta_j \rightsquigarrow N(0,1)$.

(xi) For a fixed real θ and a fixed n, define $W_k := \sum_{j\le k} w_j$ and

$$\lambda_k := \exp\left(i\theta \sum\nolimits_{j\le k} \eta_j + \tfrac{1}{2}\theta^2 W_k\right)$$

with $\lambda_0 \equiv 1$. Show that

$$\mathbb{P}_{k-1}\lambda_k = \lambda_{k-1}\exp(\tfrac{1}{2}\theta^2 w_k)\left(1 - \tfrac{1}{2}\theta^2 w_k + R_2(\theta\eta_k)\right)$$

(xii) Deduce that

$$|\mathbb{P}\lambda_k - \mathbb{P}\lambda_{k-1}| \le C_\theta \mathbb{P}\left(|\eta_k|^3 + w_k^2\right) \le C_\theta\left(n^{-1} + n^{-2}\right)\mathbb{P}w_k,$$

for some constant C_θ that depends on θ.

(xiii) Deduce that $\mathbb{P}\lambda_{k(n)} = 1 + o(1)$.

(xiv) Show that

$$|\mathbb{P}\lambda_{k(n)} - \mathbb{P}\exp\left(i\theta \sum\nolimits_{j\le k} \eta_j + \tfrac{1}{2}\theta^2\right)| \le \mathbb{P}|\exp\left(\tfrac{1}{2}\theta^2 W_{k(n)}\right) - \exp\left(\tfrac{1}{2}\theta^2\right)|,$$

which tends to 0 as $n \to \infty$.

(xv) Complete the argument that $M_1 \sim N(0,1)$.

See Pollard (2001, Section 9.6) for a slightly different implementation of the same idea for proving $M_1 \sim N(0,1)$.

Higher order fidis (for enthusiasts).

To complete the proof that M is a Brownian motion, you would need check the other fidis. For example, if $0 < s < 1$ you would need to show M_s and $M_1 - M_s$ are independent with $N(0,s)$ and $N(0,1-s)$ distributions. Equivalently, you could show that

$$\mathbb{P}\exp\left(i\alpha M_s + i\beta(M_1 - M_s)\right) = \exp\left(-\tfrac{1}{2}\alpha^2 s - \tfrac{1}{2}\beta^2(1-s)\right) \qquad \text{for all } \alpha, \beta \in \mathbb{R}.$$

- Adapt the argument from (i) to (xv) to prove the last displayed equality. Define $\gamma_n := \max\{j : W_j \le s\}$. Try working with a sum of increments $\alpha\eta_j\{j \le \gamma_n\} + \beta\eta_j\{\gamma_n < j\}$.
- Extend the argument to joint distributions of more than two increments of M.

5.2 Problems

[1] Suppose $\{X_t : t \in \mathbb{R}+\}$ is a progressively measurable process for some standard filtration $\{\mathcal{F}_t : t \in \mathbb{R}^+\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose σ is a stopping time for the filtration. Define

$$\tau(\omega) = \inf\{t \geq \sigma(\omega) : |X_t(\omega) - X_{\sigma(\omega)}| \in B\}$$

for some $B \in \mathcal{B}(\mathbb{R})$. Interpret the definition to mean that $\tau(\omega) = +\infty$ when $\sigma(\omega) = +\infty$. Show that τ is a stopping time by the following steps.

(i) Define

$$[\![\sigma, \infty[\![:= \{(t, \omega) \in \mathbb{R}^+ \times \Omega : \sigma(\omega) \leq t < \infty\}.$$

Show that this ($\{0, 1\}$-valued) process is adapted and has cadlag sample paths.

(ii) Show that the process $Y(t, \omega) := X(t \wedge \sigma(\omega), \omega)$ is progressively measurable.

(iii) Show that the set

$$A_t := \{(s, \omega) : X(s, \omega) - Y(s, \omega) \in B,\ 0 \leq s < t\} \cap [\![\sigma, \infty[\![$$

belongs to $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_t$.

(iv) Show that $\{\omega : \tau(\omega) < t\}$ equals the projection of A_t onto Ω. See Section 5 of Project 4 for the necessary measure theory.

References

Lévy, P. (1948). *Processus stochastiques et mouvement brownien*. Paris: Gauthier-Villars. Second edition, 1965.

Pollard, D. (1984). *Convergence of Stochastic Processes*. New York: Springer.

Pollard, D. (2001). *A User's Guide to Measure Theoretic Probability*. Cambridge University Press.